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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU IX

By Marin Chirciu-Romania

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1) In ΔABC the following relationship holds:

$$\sum m_a r_a \le \frac{3R}{2r}(2R^2+r^2)$$

Proposed by Marian Ursărescu-Romania

Solution by Marin Chirciu-Romania

Lemma. 2) In ΔABC the following relationship holds:

$$\sum m_a r_a \le 8R^2-3Rr+r^2$$

Proof. Using well-known identities:

$$\sum m_a^2=\frac{3}{4}\sum a^2;\ \sum a^2=2(s^2-r^2-4Rr),$$

$$\sum m_a^2=\frac{3}{2}(s^2-r^2-4Rr), \sum r_a^2=(4R+r)^2-2s^2$$

From AM-GM, we have: $m_a r_a \le \frac{m_a^2+r_a^2}{2} \Rightarrow$

$$\sum m_a r_a \le \sum \frac{m_a^2+r_a^2}{2}=\frac{1}{2}\left(\sum m_a^2+\sum r_a^2\right)=$$

$$=\frac{1}{2}\left[\frac{3}{2}(s^2-r^2-4Rr)+(4R+r)^2-2s^2\right]=\frac{1}{4}(32R^2+4Rr-r^2-s^2)\le$$

$$\overset{Gerretsen}{\le}\frac{1}{4}32R^2+4Rr-r^2-16Rr+5r^2)=\frac{1}{4}(32R^2-12Rr+4r^2)=$$

$=8R^2-3Rr+r^2$.Let's get back to the main problem.

Using Lemma, it is suffices to prove that:

$$8R^2-3Rr+r^2\le\frac{3R}{2r}(2R^2+r^2)\Leftrightarrow 2r(8R^2-3Rr+r^2)\le 3R(2R^2+r^2)$$

$\Leftrightarrow 6R^3-16R^2r+9Rr^2-2r^3\ge 0 \Leftrightarrow (R-2r)(6R^2-4Rr+r^2)\ge 0$ which is true from $R\ge 2r(Euler)$. Remark. The problem it can be developed.

3) In ΔABC the following relationship holds:

$$\sum m_a r_a \le \frac{1}{3}(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Triplets $(m_a, m_b, m_c), (r_a, r_b, r_c)$ are reverse ordered, then applying Chebyshev's inequality, we get:

$$\sum m_a r_a \le \frac{1}{3}\sum m_a \sum r_a \le \frac{1}{3}(4R+r)(4R+r) = \frac{1}{3}(4R+r)^2$$

which follows from $\sum m_a \le 4R+r$ and $\sum r_a \le 4R+r$.

Equality holds if and only if triangle is equilateral.

4) In ΔABC the following relationship holds:

$$\sum m_a r_a \le \frac{1}{3}(4R+r)^2 \le 8R^2 - 3Rr + r^2 \le \frac{3R}{2r}(2R^2+r^2)$$

Proposed by Marin Chirciu-Romania

Solution by proposer See inequality (3), $\frac{1}{3}(4R+r)^2 \overset{(1)}{\le} 8R^2 - 3Rr + r^2$ and $8R^2 - 3Rr + r^2 \overset{(2)}{\le} \frac{3R}{2r}(2R^2+r^2)$, where

$$(1) \Leftrightarrow \frac{1}{3}(4R+r)^2 \le 8R^2 - 3Rr + r^2 \Leftrightarrow (4R+r)^2 \le 3(8R^2 - 3Rr + r^2) \Leftrightarrow$$

$8R^2 - 17Rr + 2r^2 \ge 0 \Leftrightarrow (R-2r)(8R-r) \ge 0$, which is true from $R \ge 2r (Euler)$.

$(2) \Leftrightarrow 2r(8R^2 - 3Rr + r^2) \le 3R(2R^2+r^2) \Leftrightarrow (R-2r)(6R^2 - 4Rr + r^2) \ge 0$, which is true from $R \ge 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\sum m_a r_a \ge 27r^2$$

Proposed by Marin Chirciu-Romania

Solution by proposer Using AM-GM inequality: $\prod m_a \ge \prod r_a \ge 27r^3$, we get:

$$\sum m_a r_a \ge 3\sqrt[3]{\prod m_a \prod r_a} \ge 3\sqrt[3]{27r^3 \cdot 27R^3} = 3 \cdot 3r \cdot 3r = 27r^2$$

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$27r^2 \le \sum m_a r_a \le \frac{1}{3}(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution by proposer For LHS, we have:

Using AM-GM inequality: $\prod m_a \geq \prod r_a \geq 27r^3$, we get:

$$\sum m_a r_a \geq 3\sqrt[3]{\prod m_a \prod r_a} \geq 3\sqrt[3]{27r^3 \cdot 27R^3} = 3 \cdot 3r \cdot 3r = 27r^2$$

Equality holds if and only if triangle is equilateral. For RHS, we have:

Triplets $(m_a, m_b, m_c), (r_a, r_b, r_c)$ are reverse ordered, then applying Chebyshev's inequality, we get:

$$\sum m_a r_a \leq \frac{1}{3}\sum m_a \sum r_a \leq \frac{1}{3}(4R+r)(4R+r) = \frac{1}{3}(4R+r)^2$$

which follows from $\sum m_a \leq 4R + r$ and $\sum r_a \leq 4R + r$.

Equality holds if and only if triangle is equilateral.

Reference:

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