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ABOUT AN INEQUALITY BY HOANG LE NHAT TUNG-I

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

1.Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$. Find the minimum value of expression

$$P=\frac{a}{b(b+2c)(a+3c)^2}+\frac{b}{c(c+2a)(b+3a)^2}+\frac{c}{a(a+2b)(c+3b)^2}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by Marin Chirciu-Romania

$$P=\sum_{cyc}\frac{a}{b(b+2c)(a+3c)^2}=\sum_{cyc}\frac{\left(\frac{a}{a+3c}\right)^2}{ab(b+2c)}\overset{BCS}{\geq}\frac{\left(\sum_{cyc}\frac{a}{a+3c}\right)^2}{\sum_{cyc}ab(b+2c)}\overset{(1)}{\geq}$$

$$\geq\frac{\left(\frac{3}{4}\right)^2}{\sum_{cyc}a^2b+6abc}\overset{(2)}{\geq}\frac{\frac{9}{16}}{\frac{9}{8}}=\frac{1}{2}.$$ Let's proof inequality (1): $\sum_{cyc}\frac{a}{a+3c}\geq\frac{3}{4}$

$$\sum_{cyc}\frac{a}{a+3c}=\sum_{cyc}\frac{a^2}{a^2+3ac}\overset{BCS}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(a^2+3ac)}=\frac{\sum_{cyc}a^2+2\sum_{cyc}bc}{\sum_{cyc}a^2+3\sum_{cyc}bc}\overset{(3)}{\geq}\frac{3}{4}$$

Where $(3)\Leftrightarrow\sum_{cyc}a^2\geq\sum_{cyc}bc\Leftrightarrow\sum_{cyc}(b-c)^2\geq 0$ true, with equality when $a=b=c$.

Let's proof inequality (2): $\sum_{cyc}a^2b+6abc\leq\frac{9}{8}$

From AM-GM we have: $a+b\geq 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a)\geq 8abc$ and from $(a+b)(b+c)(c+a)=1$ we get $abc\leq\frac{1}{8}$; (4) with equality for $a=b=c=\frac{1}{2}$.Now,

$$(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc=1+abc\overset{(4)}{\leq}1+\frac{1}{8}=\frac{9}{8}$$

$$(a+b+c)(ab+bc+ca)\leq\frac{9}{8}\Leftrightarrow\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\leq\frac{9}{8};\quad(5)$$

Applying again AM-GM, we have $\sum_{cyc}a^2b\geq 3abc$, then from (5) it follows that

$$\frac{9}{8}\geq\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\geq\sum_{cyc}ab^2+3abc+3abc=\sum_{cyc}ab^2+6abc$$

Hence $\sum_{cyc}ab^2+6abc\leq\frac{9}{8}$. From (1),(2) it follows that $P\geq\frac{1}{2}$.

So, minimum of expression $P=\sum_{cyc}\frac{a}{b(b+2c)(a+3c)^2}$ is $\frac{1}{2}$ attained for $a=b=c=\frac{1}{2}$.

2. Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$ and $\lambda\geq 2$. Find the minimum value of expression

$$P=\frac{a}{b(b+2c)(a+\lambda c)^2}+\frac{b}{c(c+2a)(b+\lambda a)^2}+\frac{c}{a(a+2b)(c+\lambda b)^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

$$P=\sum_{cyc}\frac{a}{b(b+2c)(a+\lambda c)^2}=\sum_{cyc}\frac{\left(\frac{a}{a+\lambda c}\right)^2}{ab(b+2c)}\overset{BCS}{\geq}\frac{\left(\sum_{cyc}\frac{a}{a+\lambda c}\right)^2}{\sum_{cyc}ab(b+2c)}\overset{(1)}{\geq}$$

$$\geq\frac{\left(\frac{3}{\lambda+1}\right)^2}{\sum_{cyc}a^2b+6abc}\overset{(2)}{\geq}\frac{\frac{9}{(\lambda+1)^2}}{\frac{9}{8}}=\frac{8}{(\lambda+1)^2}.$$

Let's proof inequality (1): $\sum_{cyc}\frac{a}{a+\lambda c}\geq\frac{3}{\lambda+1}$

$$\sum_{cyc}\frac{a}{a+\lambda c}=\sum_{cyc}\frac{a^2}{a^2+\lambda ac}\overset{BCS}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(a^2+\lambda ac)}=\frac{\sum_{cyc}a^2+2\sum_{cyc}bc}{\sum_{cyc}a^2+\lambda\sum_{cyc}bc}\overset{(3)}{\geq}\frac{3}{4}$$

Where (3) $\Leftrightarrow(\lambda-2)\sum_{cyc}a^2\geq(\lambda-2)\sum_{cyc}bc\Leftrightarrow(\lambda-2)\sum_{cyc}(b-c)^2\geq 0$ true for $\lambda\geq 2$ and $\sum_{cyc}(b-c)^2\geq 0$, with equality when $a=b=c$.

Let's proof inequality (2): $\sum_{cyc}a^2b+6abc\leq\frac{9}{8}$

From AM-GM we have: $a+b\geq 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a)\geq 8abc$ and from $(a+b)(b+c)(c+a)=1$ we get $abc\leq\frac{1}{8}$; (4) with equality for $a=b=c=\frac{1}{2}$.Now,

$$(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc=1+abc\overset{(4)}{\leq}1+\frac{1}{8}=\frac{9}{8}$$

$$(a+b+c)(ab+bc+ca)\leq\frac{9}{8}\Leftrightarrow\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\leq\frac{9}{8};\ \ (5)$$

Applying again AM-GM, we have $\sum_{cyc}a^2b\geq 3abc$, then from (5) it follows that

$$\frac{9}{8}\geq\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\geq\sum_{cyc}ab^2+3abc+3abc=\sum_{cyc}ab^2+6abc$$

Hence $\sum_{cyc}ab^2+6abc\leq\frac{9}{8}$. From (1),(2) it follows that $P\geq\frac{8}{(\lambda+1)^2}$.

So, minimum of expression $P=\sum_{cyc}\frac{a}{b(b+2c)(a+\lambda c)^2}$ is $\frac{8}{(\lambda+1)^2}$ attained for $a=b=c=\frac{1}{2}$.

Note: For $\lambda=2$ we get problem SP.304 from Number 22-RMM Autumn Edition 2021, proposed by Hoang Le Nhat Tung, Hanoi, Vietnam.

3. Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$ and $\lambda\ge 2,\mu\ge 2$. Find the minimum value of expression

$$P=\frac{a}{b(b+\mu c)(a+\lambda c)^2}+\frac{b}{c(c+\mu a)(b+\lambda a)^2}+\frac{c}{a(a+\mu b)(c+\lambda b)^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

$$P=\sum_{cyc}\frac{a}{b(b+\mu c)(a+\lambda c)^2}=\sum_{cyc}\frac{\left(\frac{a}{a+\lambda c}\right)^2}{ab(b+\mu c)}\overset{BCS}{\ge}\frac{\left(\sum_{cyc}\frac{a}{a+\lambda c}\right)^2}{\sum_{cyc}ab(b+\mu c)}\overset{(1)}{\ge}$$

$$\ge\frac{\left(\frac{3}{\lambda+1}\right)^2}{\sum_{cyc}a^2b+3\mu abc}\overset{(2)}{\ge}\frac{\frac{9}{(\lambda+1)^2}}{\frac{3(\mu+1)}{8}}=\frac{24}{(\mu+1)(\lambda+1)^2}.$$

Let's proof inequality (1): $\sum_{cyc}\frac{a}{a+\lambda c}\ge\frac{3}{\lambda+1}$

$$\sum_{cyc}\frac{a}{a+\lambda c}=\sum_{cyc}\frac{a^2}{a^2+\lambda ac}\overset{BCS}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(a^2+\lambda ac)}=\frac{\sum_{cyc}a^2+2\sum_{cyc}bc}{\sum_{cyc}a^2+\lambda\sum_{cyc}bc}\overset{(3)}{\ge}\frac{3}{4}$$

Where (3) $\Leftrightarrow(\lambda-2)\sum_{cyc}a^2\ge(\lambda-2)\sum_{cyc}bc\Leftrightarrow(\lambda-2)\sum_{cyc}(b-c)^2\ge 0$ true for $\lambda\ge 2$ and $\sum_{cyc}(b-c)^2\ge 0$, with equality when $a=b=c$.

Let's proof inequality (2): $\sum_{cyc}a^2b+3\mu abc\le\frac{3(\mu+1)}{8}$

From AM-GM we have: $a+b\ge 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a)\ge 8abc$ and from $(a+b)(b+c)(c+a)=1$ we get $abc\le\frac{1}{8}$; (4) with equality for $a=b=c=\frac{1}{2}$. Now,

$$(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc=1+abc\overset{(4)}{\le}1+\frac{1}{8}=\frac{9}{8}$$

$$(a+b+c)(ab+bc+ca)\le\frac{9}{8}\Leftrightarrow\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\le\frac{9}{8};\quad(5)$$

Applying again AM-GM, we have $\sum_{cyc}a^2b\ge 3abc$, then from (5) it follows that

$$\frac{9}{8}\ge\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\ge\sum_{cyc}ab^2+3abc+3abc=\sum_{cyc}ab^2+6abc$$

Hence $\sum_{cyc}ab^2+6abc\le\frac{9}{8}$; (6)

$\sum_{cyc}a^2b+3\mu abc\le\frac{3(\mu+1)}{8}$ true from (6) and (4), for $\mu\ge 2$ we get (2).

From (1),(2) it follows that $P\ge\frac{24}{(\mu+1)(\lambda+1)^2}$.

So, minimum of expression $P=\sum_{cyc}\frac{a}{b(b+\mu c)(a+\lambda c)^2}$ is $\frac{24}{(\mu+1)(\lambda+1)^2}$ attained for

$$a = b = c = \frac{1}{2}.$$

4. Let $a, b, c > 0$ such that $(a+b)(b+c)(c+a) = 1$ and $\lambda \geq 2, n \in \mathbb{N}^*$.

Find the minimum value of expression

$$P = \frac{a^n}{b(b+2c)(a+\lambda c)^{n+1}} + \frac{b^n}{c(c+2a)(b+\lambda a)^{n+1}} + \frac{c^n}{a(a+2b)(c+\lambda b)^{n+1}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

$$P = \sum_{cyc} \frac{a^n}{b(b+2c)(a+\lambda c)^{n+1}} = \sum_{cyc} \frac{\left(\frac{a}{a+\lambda c}\right)^{n+1}}{ab(b+2c)} \overset{Holder}{\geq} \frac{\left(\sum_{cyc} \frac{a}{a+\lambda c}\right)^{n+1}}{3^{n-1}\sum_{cyc} ab(b+2c)} \overset{(1)}{\geq}$$

$$\geq \frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{3^{n-1}\sum_{cyc} a^2b + 6abc} \overset{(2)}{\geq} \frac{\frac{9}{(\lambda+1)^{n+1}}}{\frac{9}{8}} = \frac{8}{(\lambda+1)^{n+1}}.$$

Let's proof inequality (1): $\sum_{cyc} \frac{a}{a+\lambda c} \geq \frac{3}{\lambda+1}$

$$\sum_{cyc} \frac{a}{a+\lambda c} = \sum_{cyc} \frac{a^2}{a^2+\lambda ac} \overset{BCS}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc}(a^2+\lambda ac)} = \frac{\sum_{cyc} a^2 + 2\sum_{cyc} bc}{\sum_{cyc} a^2 + \lambda \sum_{cyc} bc} \overset{(3)}{\geq} \frac{3}{4}$$

Where (3) $\Leftrightarrow (\lambda-2)\sum_{cyc} a^2 \geq (\lambda-2)\sum_{cyc} bc \Leftrightarrow (\lambda-2)\sum_{cyc}(b-c)^2 \geq 0$ true for $\lambda \geq 2$ and $\sum_{cyc}(b-c)^2 \geq 0$, with equality when $a = b = c$.

Let's proof inequality (2): $\sum_{cyc} a^2b + 6abc \leq \frac{9}{8}$

From AM-GM we have: $a + b \geq 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a) \geq 8abc$ and from $(a+b)(b+c)(c+a) = 1$ we get $abc \leq \frac{1}{8}$; (4) with equality for $a = b = c = \frac{1}{2}$. Now,

$$(a+b+c)(ab+bc+ca) = (a+b)(b+c)(c+a) + abc = 1 + abc \overset{(4)}{\leq} 1 + \frac{1}{8} = \frac{9}{8}$$

$$(a+b+c)(ab+bc+ca) \leq \frac{9}{8} \Leftrightarrow \sum_{cyc} ab^2 + \sum_{cyc} a^2b + 3abc \leq \frac{9}{8}; \quad (5)$$

Applying again AM-GM, we have $\sum_{cyc} a^2b \geq 3abc$, then from (5) it follows that

$$\frac{9}{8} \geq \sum_{cyc} ab^2 + \sum_{cyc} a^2b + 3abc \geq \sum_{cyc} ab^2 + 3abc + 3abc = \sum_{cyc} ab^2 + 6abc$$

Hence $\sum_{cyc} ab^2 + 6abc \leq \frac{9}{8}$. From (1),(2) it follows that $P \geq \frac{8}{(\lambda+1)^{n+1}}$.

So, minimum of expression $P = \sum_{cyc} \frac{a}{b(b+2c)(a+\lambda c)^2}$ is $\frac{8}{(\lambda+1)^{n+1}}$ attained for $a = b = c = \frac{1}{2}$.

Note. For $n=1,\lambda=2$ we get problem SP.304 from Number 22-RMM Autumn Edition 2021, proposed by Hoang Le Nhat Tung-Hanoi-Vietnam.

5. Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$ and $\lambda\ge 2,\mu\ge 2$. Find the minimum value of expression

$$P=\frac{a^n}{b(b+\mu c)(a+\lambda c)^{n+1}}+\frac{b^n}{c(c+\mu a)(b+\lambda a)^{n+1}}+\frac{c^n}{a(a+\mu b)(c+\lambda b)^{n+1}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

$$P=\sum_{cyc}\frac{a^n}{b(b+\mu c)(a+\lambda c)^{n+1}}=\sum_{cyc}\frac{\left(\frac{a}{a+\lambda c}\right)^{n+1}}{ab(b+\mu c)}\overset{Holder}{\ge}\frac{\left(\sum_{cyc}\frac{a}{a+\lambda c}\right)^{n+1}}{3^{n-1}\sum_{cyc}ab(b+\mu c)}\overset{(1)}{\ge}$$

$$\ge\frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{3^{n-1}\sum_{cyc}a^2b+3\mu abc}\overset{(2)}{\ge}\frac{\frac{9}{(\lambda+1)^{n+1}}}{\frac{3(\mu+1)}{8}}=\frac{24}{(\mu+1)(\lambda+1)^{n+1}}.$$

Let's proof inequality (1): $\sum_{cyc}\frac{a}{a+\lambda c}\ge\frac{3}{\lambda+1}$

$$\sum_{cyc}\frac{a}{a+\lambda c}=\sum_{cyc}\frac{a^2}{a^2+\lambda ac}\overset{BCS}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(a^2+\lambda ac)}=\frac{\sum_{cyc}a^2+2\sum_{cyc}bc}{\sum_{cyc}a^2+\lambda\sum_{cyc}bc}\overset{(3)}{\ge}\frac{3}{4}$$

Where (3) $\Leftrightarrow(\lambda-2)\sum_{cyc}a^2\ge(\lambda-2)\sum_{cyc}bc\Leftrightarrow(\lambda-2)\sum_{cyc}(b-c)^2\ge 0$ true for $\lambda\ge 2$ and $\sum_{cyc}(b-c)^2\ge 0$, with equality when $a=b=c$.

Let's proof inequality (2): $\sum_{cyc}a^2b+3\mu abc\le\frac{3(\mu+1)}{8}$

From AM-GM we have: $a+b\ge 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a)\ge 8abc$ and from $(a+b)(b+c)(c+a)=1$ we get $abc\le\frac{1}{8}$; (4) with equality for $a=b=c=\frac{1}{2}$. Now,

$$(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc=1+abc\overset{(4)}{\le}1+\frac{1}{8}=\frac{9}{8}$$

$$(a+b+c)(ab+bc+ca)\le\frac{9}{8}\Leftrightarrow\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\le\frac{9}{8};\quad(5)$$

Applying again AM-GM, we have $\sum_{cyc}a^2b\ge 3abc$, then from (5) it follows that

$$\frac{9}{8}\ge\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\ge\sum_{cyc}ab^2+3abc+3abc=\sum_{cyc}ab^2+6abc$$

Hence $\sum_{cyc}ab^2+6abc\le\frac{9}{8}$; (6)

$\sum_{cyc}a^2b+3\mu abc\le\frac{3(\mu+1)}{8}$ true from (6) and (4), for $\mu\ge 2$ we get (2).

From (1),(2) it follows that $P \geq \frac{24}{(\mu+1)(\lambda+1)^{n+1}}$.

So, minimum of expression $P = \sum_{cyc} \frac{a^n}{b(b+\mu c)(a+\lambda c)^{n+1}}$ is $\frac{24}{(\mu+1)(\lambda+1)^{n+1}}$ attained for $a = b = c = \frac{1}{2}$.

Note. For $n = 1, \lambda = 3, \mu = 2$ we get problem SP.304 from RMM Autumn Edition 2021, proposed by Hoang Le Nhat-Hanoi-Vietnam

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