ABOUT AN INEQUALITY BY ERTAN YILDIRIM-VI

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1) In ΔABC he following relationship holds:

$$72r^2 \le \sum \frac{w_b^2 + w_c^2}{sinBsinC} \le \frac{9R^3}{r}$$

Proposed by Ertan Yildirim-Turkey

Solution. Using law of sines: $sinA = \frac{a}{2R}$, we have:

$$72r^2 \le \sum \frac{w_b^2 + w_c^2}{sinBsinC} \le \frac{9R^3}{r} \Leftrightarrow 72r^2 \le \sum \frac{w_b^2 + w_c^2}{\frac{b}{2R}\frac{c}{2R}} \le \frac{9R^3}{r} \Leftrightarrow$$

$$\frac{72r^2}{4R^2} \le \sum \frac{w_b^2 + w_c^2}{bc} \le \frac{9R^3}{4R^2 r} \Leftrightarrow \frac{18r^2}{R^2} \le \sum \frac{w_b^2 + w_c^2}{bc} \le \frac{9R}{4r}$$

Lemma. 2) In ΔABC the following relationship holds:

$$\sum \frac{h_b^2 + h_c^2}{bc} = \frac{s^2(s^2 + 2r^2 - 10Rr) + r^2(8R^2 + 6Rr + r^2)}{8R^3 r}$$

For LHS, using Lemma and $w_a \ge h_a$, it follows:

$$\sum \frac{w_b^2 + w_c^2}{bc} \ge \sum \frac{h_b^2 + h_c^2}{bc} = \frac{s^2(s^2 + 2r^2 - 10Rr) + r^2(8R^2 + 6Rr + r^2)}{8R^3 r} \overset{Gerretsen}{\ge}$$

$$\ge \frac{(16Rr - 5r^2)(16Rr - 5r^2 + 2r^2 - 10Rr) + r^2(8R^2 + 6Rr + r^2)}{8R^3} =$$

$$= \frac{r(96R^2 - 48Rr - 30Rr + 15r^2 + 8R^2 + 6Rr + r^2)}{8R^3} = \frac{r(104R^2 - 72Rr + 16r^2)}{8R^3} =$$

$$= \frac{r(13R^2 - 9Rr + 2r^2)}{R^3} = \frac{r}{R}\left(13 - \frac{9r}{R} + \frac{2r^2}{R^2}\right)$$

We must to prove that:

$$\frac{r(13R^2 - 9Rr + 2r^2)}{R^3} \ge \frac{18r^2}{R^2} \Leftrightarrow 13R^2 - 9Rr + 2r^2 \ge 18Rr \Leftrightarrow$$

$13R^2 - 27Rr + 2r^2 \ge 0 \Leftrightarrow (R - 2r)(13R - r) \ge 0$, which is true from $R \ge 2r(Euler)$.

For RHS, using $w_a \le \sqrt{s(s-a)}$, it follows that:

$$\sum\frac{w_b^2+w_c^2}{bc}\le\sum\frac{s(s-b)+s(s-c)}{bc}=\sum\frac{sa}{bc}=s\sum\frac{a}{bc}=\frac{s\sum a^2}{abc}=$$

$$=\frac{2s(s^2-r^2-4Rr)}{4Rrs}=\frac{s^2}{2Rr}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2-r^2-4Rr}{2Rr}=$$

$$=\frac{2R^2+r^2}{Rr}=\frac{2R}{r}+\frac{r}{R}$$

Remains to prove that:

$\frac{2R^2+r^2}{Rr}\le\frac{9R}{4r}\Leftrightarrow 4(2R^2+r^2)\le 9R^2\Leftrightarrow R^2\ge 4r^2$**, which is true from** $R\ge 2r(Euler)$**.**

Remark. Inequality can be much stronger.

3) In ΔABC the following relationship holds:

$$\frac{r}{R}\left(13-\frac{9r}{R}+\frac{2r^2}{R^2}\right)\le\sum\frac{w_b^2+w_c^2}{bc}\le\frac{2R}{r}+\frac{r}{R}$$

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Solution.

Lemma. 2) In ΔABC the following relationship holds:

$$\sum\frac{h_b^2+h_c^2}{bc}=\frac{s^2(s^2+2r^2-10Rr)+r^2(8R^2+6Rr+r^2)}{8R^3r}$$

Using $w_a\ge h_a$and Lemma, it follows that:

Lemma. 2) In ΔABC the following relationship holds:

$$\sum\frac{h_b^2+h_c^2}{bc}=\frac{s^2(s^2+2r^2-10Rr)+r^2(8R^2+6Rr+r^2)}{8R^3r}\overset{Gerretsen}{\ge}$$

$$\ge\frac{(16Rr-5r^2)(16Rr-5r^2+2r^2-10Rr)+r^2(8R^2+6Rr+r^2)}{8R^3}$$

$$=\frac{r(96R^2-48Rr-30Rr+15r^2+8R^2+6Rr+r^2)}{8R^3}=\frac{r(104R^2-72Rr+16r^2)}{8R^3}=$$

$$=\frac{r(13R^2-9Rr+2r^2)}{R^3}=\frac{r}{R}\left(13-\frac{9r}{R}+\frac{2r^2}{R^2}\right)$$

Equality holds if and only if triangle is equilateral.

For RHS, using $w_a\le\sqrt{s(s-a)}$, it follows that:

$$\sum\frac{w_b^2+w_c^2}{bc}\le\sum\frac{s(s-b)+s(s-c)}{bc}=\sum\frac{sa}{bc}=s\sum\frac{a}{bc}=\frac{s\sum a^2}{abc}=$$

$$=\frac{2s(s^2-r^2-4Rr)}{4Rrs}=\frac{s^2}{2Rr}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2-r^2-4Rr}{2Rr}=$$

$$=\frac{2R^2+r^2}{Rr}=\frac{2R}{r}+\frac{r}{R}$$

Remains to prove that:

$\frac{2R^2+r^2}{Rr}\le\frac{9R}{4r}\Leftrightarrow 4(2R^2+r^2)\le 9R^2\Leftrightarrow R^2\ge 4r^2$, **which is true from** $R\ge 2r(Euler)$.

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References:

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