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ABOUT AN INEQUALITY BY ALEX SZOROS

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1) In ΔABC the following relationship holds:

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\ge\frac{2abc}{R}$$

Proposed by Alex Szoros-Romania

Solution. **2) Lemma. In ΔABC the following relationship holds:**

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}=\frac{2[s^2(2R+3r)-r(4R+r)^2]}{s}$$

Proof. Using identity $r_a=\frac{F}{s-a}$, we get:

$$\sum_{cyc}\frac{a^3}{r_a}=\sum_{cyc}\frac{a^3}{\frac{F}{s-a}}=\frac{1}{F}\sum_{cyc}a^3(s-a)=\frac{1}{rs}\cdot 2r[s^2(2R+3r)-r(4R+r)^2]=$$

$=\frac{2[s^2(2R+3r)-r(4R+r)^2]}{s}$, which follows from

$$\sum_{cyc}a^3(s-a)=2r[s^2(2R+3r)-r(4R+r)^2]$$

Let's get back to the main problem. Using Lemma and $abc=4Rrs$, inequality can be written as:

$$\frac{2[s^2(2R+3r)-r(4R+r)^2]}{s}\ge\frac{2\cdot 4Rrs}{R}\Leftrightarrow s^2(2R+3r)-r(4R+r)^2\ge 4rs^2$$

$\Leftrightarrow s^2(2R+3r)-r(4R+r)^2\ge 4rs^2\Leftrightarrow s^2(2R+3r)\ge r(4R+r)^2$, which follows from

$s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}(Gerretsen)$. Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}(2R-r)\ge r(4R+r)^2\Leftrightarrow 2R-r\ge R+r\Leftrightarrow R\ge 2r(Euler).$$

Equality holds if and only if triangle is equilateral.**Remark.** Inequality it can be much stronger.

3) In ΔABC the following relationship holds:

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\ge\frac{4r}{s}(8R^2+15Rr-8r^2)$$

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Solution. **Using Lemma and $s^2 \ge 16Rr - 5r^2 (Gerretsen)$, we get:**

$$\sum_{cyc} \frac{a^3}{r_a} = \frac{2[s^2(2R+3r) - r(4R+r)^2]}{s} \ge \frac{2[(16Rr - 5r^2)(2R+3r) - r(4R+r)^2]}{s} =$$

$$= \frac{4r}{s}(8R^2 + 15Rr - 8r^2).$$ Equality holds if and only if triangle is equilateral.

4) In ΔABC the following relationship holds:

$$\frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \ge \frac{4r}{s}(8R^2 + 15Rr - 8r^2) \ge \frac{2abc}{R}$$

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Solution. See up these inequalities and $\frac{4r}{s}(8R^2 + 15Rr - 8r^2) \ge \frac{2abc}{R} \Leftrightarrow$

$\frac{4r}{s}(8R^2 + 15Rr - 8r^2) \ge \frac{2\cdot 4Rrs}{R} \Leftrightarrow 8R^2 + 15Rr - 8r^2 \ge 2s^2$, which follows from

$s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$ and $R \ge 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \ge \frac{4r}{s}(8R^2 + 15Rr - 8r^2) \ge \frac{2r(4R+r)^2(R+2r)}{s(R+r)} \ge \frac{2abc}{R}$$

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Solution. See inequality $\frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \ge \frac{4r}{s}(8R^2 + 15Rr - 8r^2)$ and

$$\frac{4r}{s}(8R^2 + 15Rr - 8r^2) \ge \frac{2r(4R+r)^2(R+2r)}{s(R+r)} \Leftrightarrow$$

$$2(R+r)(8R^2 + 15Rr - 8r^2) \ge (4R+r)^2(R+2r) \Leftrightarrow R \ge 2r (Euler).$$

$$\frac{2r(4R+r)^2(R+2r)}{s(R+r)} \ge \frac{2abc}{R} \Leftrightarrow \frac{2r(4R+r)^2(R+2r)}{s(R+r)} \ge \frac{2\cdot 4Rrs}{R} \Leftrightarrow$$

$(4R+r)^2(R+2r) \ge 4s^2(R+r)$, which follows from $s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$ and $R \ge 2r (Euler)$.Equality holds if and only if triangle is equilateral.

Remark. Let's find an reverse inequality.

6) In ΔABC the following relationship holds:

$$\frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \le \frac{(4R+r)^2(2R^2 - Rr + 2r^2)}{s(2R-r)}$$

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Solution. Using Lemma and $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2 + 4Rr + 3r^2 (Blundon\ Gerretsen)$, we get: $\sum_{cyc} \frac{a^3}{r_a} = \frac{2[s^2(2R+3r)-r(4R+r)^2]}{s} \leq \frac{2\left[\frac{R(4R+r)^2}{2(2R-2)}(2R+3r)-r(4R+r)^2\right]}{s} =$

$= \frac{2(4R+r)^2[R(2R+3r)-2r(2R-r)]}{2s(2R-r)} = \frac{(4R+r)^2(2R^2-Rr+2r^2)}{s(2R-r)}$, equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$\frac{(4R+r)^2}{s} \cdot \frac{2r(R+2r)}{R+r} \leq \frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \leq \frac{(4R+r)^2}{s} \cdot \frac{2R^2-Rr+2r^2}{2R-r}$$

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Solution. For LHS, using Lemma and $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r} (Gerretsen)$, we get:

$$\sum_{cyc} \frac{a^3}{r_a} = \frac{2[s^2(2R+3r)-r(4R+r)^2]}{s} \geq \frac{2\left[\frac{r(4R+r)^2}{R+r}(2R+3r)-r(4R+r)^2\right]}{s} =$$

$$= \frac{2r(4R+r)^2(R+2r)}{s(R+r)}$$

For RHS, using Lemma and $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2 + 4Rr + 3r^2 (Blundon\ Gerretsen)$, we get: $\sum_{cyc} \frac{a^3}{r_a} = \frac{2[s^2(2R+3r)-r(4R+r)^2]}{s} \leq \frac{2\left[\frac{R(4R+r)^2}{2(2R-r)}(2R+3r)-r(4R+r)^2\right]}{s} =$

$$= \frac{2(4R+r)^2[R(2R+3r)-2r(2R-r)]}{2s(2R-r)} = \frac{(4R+r)^2(2R^2-Rr+2r^2)}{s(2R-r)}$$

Equality holds if and only if triangle is equilateral.

8) In ΔABC the following relationship holds:

$$\frac{(4R+r)^2}{s} \cdot \frac{2r(R+2r)}{R+r} \leq \frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c} \leq \frac{(4R+r)^2}{s} \cdot \frac{2R^2-Rr+2r^2}{2R-r} \leq \frac{2abc}{R}$$

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Solution. See up these inequalities. Equality holds if and only if triangle is equilateral.

References:

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