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ROMANIAN MATHEMATICAL MAGAZINE

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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-XI

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\left(\frac{a}{m_a}\right)^4+\left(\frac{b}{m_b}\right)^4+\left(\frac{c}{m_c}\right)^4\geq\frac{16}{3}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution. **Lemma. 2) In ΔABC the following relationship hols:**

$$\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\geq 2\sqrt{3}$$

Proof. Using identity: $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ and AM-GM inequality, we get:

$$LHS=\sum_{cyc}\frac{a}{m_a}=\sum_{cyc}\frac{2a}{\sqrt{2b^2+2c^2-a^2}}=\sum_{cyc}\frac{2\sqrt{3}\cdot a^2}{\sqrt{(2b^2+2c^2-a^2)3a^2}}\geq$$

$$\geq\sum_{cyc}\frac{2\sqrt{3}\cdot a^2}{\frac{(2b^2+2c^2-a^2)+3a^2}{2}}=2\sqrt{3}\sum_{cyc}\frac{a^2}{b^2+c^2+a^2}=2\sqrt{3}$$

With equality if $2b^2+2c^2-a^2=3a^2$ and analogs. Equality holds if and only if triangle is equilateral.Let's get back to the main problem. Using Lemma and Holder Inequality, we get:

$$LHS=\sum_{cyc}\left(\frac{a}{m_a}\right)^4\geq\frac{\left(\sum_{cyc}\frac{a}{m_a}\right)^4}{27}>-\frac{\left(2\sqrt{3}\right)^4}{27}=\frac{16}{3}=RHS$$

Equality if and only if triangle is equilateral. Remark. Inequality can be developed.

3) In ΔABC the following relationship holds:

$$\left(\frac{a}{m_a}\right)^{2n}+\left(\frac{b}{m_b}\right)^{2n}+\left(\frac{c}{m_c}\right)^{2n}\geq 3\left(\frac{4}{3}\right)^n, n\in\mathbb{N}$$

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Solution. **Lemma. 4) In ΔABC the following relationship holds:**

$$\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\geq 2\sqrt{3}$$

Proof. Using identity: $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$ and AM-GM inequality, we get:

$$LHS = \sum_{cyc}\frac{a}{m_a} = \sum_{cyc}\frac{2a}{\sqrt{2b^2+2c^2-a^2}} = \sum_{cyc}\frac{2\sqrt{3}\cdot a^2}{\sqrt{(2b^2+2c^2-a^2)3a^2}} \geq$$

$$\geq \sum_{cyc}\frac{2\sqrt{3}\cdot a^2}{\frac{(2b^2+2c^2-a^2)+3a^2}{2}} = 2\sqrt{3}\sum_{cyc}\frac{a^2}{b^2+c^2+a^2} = 2\sqrt{3}$$

With equality if $2b^2+2c^2-a^2 = 3a^2$ and analogs. Equality holds if and only if triangle is equilateral.Let's get back to the main problem. Using Lemma and Holder Inequality, we get:

$$LHS = \sum_{cyc}\left(\frac{a}{m_a}\right)^{2n} \geq \frac{\left(\sum_{cyc}\frac{a}{m_a}\right)^{2n}}{27} > -\frac{\left(2\sqrt{3}\right)^{2n}}{27} = \frac{2^{2n}\cdot 3^n}{3^{2n-1}} = \frac{2^{2n}}{3^{n-1}} = 3\left(\frac{4}{3}\right)^n = RHS$$

Equality if and only if triangle is equilateral.

Note.For $n = 2$ we obtain Inequality in triangle-1525 proposed by Adil Abdullayev-Baku-Azerbaijan in R.M.M.-no.4/2020.

References:

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