ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-VIII

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

1) In ΔABC the following relationship holds:

$$6\sqrt{r} \le \sum \frac{a}{\sqrt{h_a}} \le \frac{3R}{\sqrt{r}}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution. **For RHS inequality, using BCS inequality, we get:**

$$\left(\sum \frac{a}{\sqrt{h_a}}\right)^2 \overset{BCS}{\le} \sum a^2 \sum \frac{1}{h_a} \overset{Leibniz}{\le} 9R^2 \cdot \frac{1}{r} = \left(\frac{3R}{\sqrt{r}}\right)^2 \Rightarrow \sum \frac{a}{\sqrt{h_a}} \le \frac{3R}{\sqrt{r}}$$

Equality holds if and only if triangle is equilateral.

For LHS, using AM-GM inequality, we get:

$$\sum \frac{a}{\sqrt{h_a}} \ge 3\sqrt[3]{\prod \frac{a}{\sqrt{h_a}}} = 3\sqrt[3]{\frac{abc}{\sqrt{h_a h_b h_c}}} \overset{(1)}{\ge} 6\sqrt{r},$$

$$(1) \Leftrightarrow \sqrt[3]{\frac{abc}{\sqrt{h_a h_b h_c}}} \ge 2\sqrt{r} \Leftrightarrow \frac{(abc)^2}{h_a h_b h_c} \ge 64r^3 \Leftrightarrow \frac{16R^2r^2s^2}{\frac{2s^2r^2}{R}} \ge 64r^3 \Leftrightarrow R^3 \ge 8r^3$$

Which is true from $R \ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. Replacing h_a with r_a, we get:

2) In ΔABC the following relationship holds:

$$6\sqrt{r} \le \sum \frac{a}{\sqrt{r_a}} \le \frac{3R}{\sqrt{r}}$$

Proposed by Marin Chirciu-Romania

Solution. **For RHS inequality, using BCS inequality, we get:**

$$\left(\sum \frac{a}{\sqrt{r_a}}\right)^2 \overset{BCS}{\le} \sum a^2 \sum \frac{1}{r_a} \overset{Leibniz}{\le} 9R^2 \frac{1}{r} = \left(\frac{3R}{\sqrt{r}}\right)^2 \Rightarrow \sum \frac{a}{\sqrt{r_a}} \ge \frac{3R}{\sqrt{r}}$$

Equality holds if and only if triangle is equilateral.

For RHS, using AM-GM inequality, we get:

$$\sum \frac{a}{\sqrt{r_a}} \geq 3\sqrt[3]{\prod \frac{a}{\sqrt{r_a}}} = 3\sqrt[3]{\frac{abc}{\sqrt{r_a r_b r_c}}} \overset{(1)}{\geq} 6\sqrt{r},$$

$$(1) \Leftrightarrow \sqrt[3]{\frac{abc}{\sqrt{r_a r_b r_c}}} \geq 2\sqrt{r} \Leftrightarrow \frac{(abc)^2}{r_a r_b r_c} \geq 64r^3 \Leftrightarrow \frac{16R^2 r^2 s^2}{s^2 r} \geq 64r^3 \Leftrightarrow$$

$$R^2 \geq 4r^2 \Leftrightarrow R \geq 2r(Euler).$$

References:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro