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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-VII

By Marin Chirciu-Romania

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1.In ΔABC the following relationship holds:

$$\prod_{cyc}\frac{m_b^2+m_c^2}{m_a^2}\le\left(\frac{R}{r}\right)^3$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Marin Chirciu-Romania

Denote $m_a^2=x, m_b^2=y, m_c^2=z$ we have:

$$x+y+z=\sum_{cyc}m_a^2=\frac{3}{4}\sum_{cyc}a^2=\frac{3}{4}\cdot 2(s^2-r^2-4Rr)=\frac{3}{2}(s^2-r^2-4Rr)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\sum_{cyc}\frac{1}{m_a^2}=\frac{s^2-r^2-4Rr}{2s^2r^2}$$

It follows that:

$$LHS=\prod_{cyc}\frac{m_b^2+m_c^2}{m_a^2}=\prod_{cyc}\frac{y+z}{x}=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-1\le$$

$$\le\frac{3}{2}(s^2-r^2-4Rr)\cdot\frac{s^2-r^2-4Rr}{2s^2r^2}=\frac{3}{4s^2r^2}(s^2-r^2-4Rr)^2-1\overset{(1)}{\le}\left(\frac{R}{r}\right)^3=RHD$$

Where $(1)\Leftrightarrow\frac{3}{4s^2r^2}(s^2-r^2-4Rr)^2-1\le\left(\frac{R}{r}\right)^3\Leftrightarrow$

$$3r(s^2-r^2-4Rr)^2-4s^2r^2\le 4s^2R^3\Leftrightarrow$$

$s^2(4R^3+24Rr^2+6r^3-rs^2)\ge 3r^3(4R+r)^2$ which follows from

$$\frac{r(4R+r)^2}{R+r}\le 16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2\ (Gerretsen)$$

It is enough to prove that:

$$\frac{r(4R+r)^2}{R+r}\left(4R^3+24Rr^2+6r^3-r(4R^2+4Rr+3r^2)\right)\ge 3r^3(4R+r)^2\Leftrightarrow$$

$$4R^3-12Rr^2+12Rr+r^3\ge 3r^2(R+r)\Leftrightarrow 4R^3-12Rr^2+9rr-2r^3\ge 0\Leftrightarrow$$

$(R-2r)(2R-r)^2 \ge 0$ which is true from $R \ge 2r$ $(Euler)$

Equality holds if and only if triangle is equilateral. Remark. In same class of problem.

2. In ΔABC the following relationship holds:

$$\prod_{cyc} \frac{w_b^2 + w_c^2}{w_a^2} \le \left(\frac{R}{r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by proposer Denote $w_a^2 = x, w_b^2 = y, w_c^2 = z$ we have:

$$x+y+z = \sum_{cyc} w_a^2 \le \sum_{cyc} m_a^2 = \frac{3}{4}\sum_{cyc} a^2 = \frac{3}{4}\cdot 2(s^2-r^2-4Rr) = \frac{3}{2}(s^2-r^2-4Rr)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \sum_{cyc}\frac{1}{w_a^2} \le \sum_{cyc}\frac{1}{h_a^2} = \frac{s^2-r^2-4Rr}{2s^2r^2}$$

It follows that:

$$LHS = \prod_{cyc}\frac{w_b^2+w_c^2}{w_a^2} = \prod_{cyc}\frac{y+z}{x} = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) - 1 \le$$

$$\le \frac{3}{2}(s^2-r^2-4Rr)\cdot\frac{s^2-r^2-4Rr}{2s^2r^2} = \frac{3}{4s^2r^2}(s^2-r^2-4Rr)^2 - 1 \overset{(1)}{\le} \left(\frac{R}{r}\right)^3 = RHD$$

Where $(1) \Leftrightarrow \frac{3}{4s^2r^2}(s^2-r^2-4Rr)^2 - 1 \le \left(\frac{R}{r}\right)^3 \Leftrightarrow$

$$3r(s^2-r^2-4Rr)^2 - 4s^2r^2 \le 4s^2R^3 \Leftrightarrow$$

$s^2(4R^3+24Rr^2+6r^3-rs^2) \ge 3r^3(4R+r)^2$ which follows from

$$\frac{r(4R+r)^2}{R+r} \le 16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2 (Gerretsen)$$

It is enough to prove that:

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$(R-2r)(2R-r)^2 \ge 0$ which is true from $R \ge 2r$ $(Euler)$

Equality holds if and only if triangle is equilateral.

3. In ΔABC the following relationship holds:

$$\prod_{cyc} \frac{h_b^2 + h_c^2}{h_a^2} \le \left(\frac{R}{r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by proposer **Denote** $h_a^2 = x, h_b^2 = y, h_c^2 = z$ **we have:**

$$x + y + z = \sum_{cyc} h_a^2 \le \sum_{cyc} m_a^2 = \frac{3}{4}\sum_{cyc} a^2 = \frac{3}{4} \cdot 2(s^2 - r^2 - 4Rr) = \frac{3}{2}(s^2 - r^2 - 4Rr)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \sum_{cyc} \frac{1}{h_a^2} = \frac{s^2 - r^2 - 4Rr}{2s^2r^2}$$

It follows that:

$$LHS = \prod_{cyc} \frac{h_b^2 + h_c^2}{h_a^2} = \prod_{cyc} \frac{y + z}{x} = (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 1 \le$$

$$\le \frac{3}{2}(s^2 - r^2 - 4Rr) \cdot \frac{s^2 - r^2 - 4Rr}{2s^2r^2} = \frac{3}{4s^2r^2}(s^2 - r^2 - 4Rr)^2 - 1 \overset{(1)}{\le} \left(\frac{R}{r}\right)^3 = RHD$$

Where $(1) \Leftrightarrow \frac{3}{4s^2r^2}(s^2 - r^2 - 4Rr)^2 - 1 \le \left(\frac{R}{r}\right)^3 \Leftrightarrow$

$$3r(s^2 - r^2 - 4Rr)^2 - 4s^2r^2 \le 4s^2R^3 \Leftrightarrow$$

$s^2(4R^3 + 24Rr^2 + 6r^3 - rs^2) \ge 3r^3(4R + r)^2$ **which follows from**

$$\frac{r(4R + r)^2}{R + r} \le 16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$$

It is enough to prove that:

$$\frac{r(4R + r)^2}{R + r}\left(4R^3 + 24Rr^2 + 6r^3 - r(4R^2 + 4Rr + 3r^2)\right) \ge 3r^3(4R + r)^2 \Leftrightarrow$$

$$4R^3 - 12Rr^2 + 12Rr + r^3 \ge 3r^2(R + r) \Leftrightarrow 4R^3 - 12Rr^2 + 9rr - 2r^3 \ge 0 \Leftrightarrow$$

$(R - 2r)(2R - r)^2 \ge 0$ **which is true from** $R \ge 2r$ $(Euler)$

Equality holds if and only if triangle is equilateral.

4. In ΔABC the following relationship holds:

$$\prod_{cyc} \frac{r_b^2 + r_c^2}{r_a^2} \le \left(\frac{R}{r} - 1\right)\left(\frac{2R}{r} - 1\right)^2 - 1$$

Proposed by Marin Chirciu-Romania

Solution by proposer **Denote $r_a^2 = x, r_b^2 = y, r_c^2 = z$ we have:**

$$x + y + z = \sum_{cyc} r_a^2 = (4R + r)^2 - 2s^2$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \sum_{cyc} \frac{1}{r_a^2} = \frac{s^2 - 2r^2 - 8Rr}{s^2 r^2}$$

It follows that:

$$LHS = \prod_{cyc} \frac{r_b^2 + r_c^2}{r_a^2} = \prod_{cyc} \frac{y + z}{x} = (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 1 \le$$

$$= [(4R + r)^2 - 2s^2] \cdot \frac{s^2 - 2r^2 - 8Rr}{s^2 r^2} - 1 =$$

$$= \frac{[(4R + r)^2 - 2s^2] \cdot (s^2 - 2r^2 - 8Rr)}{s^2 r^2} - 1 =$$

$$= \frac{1}{r^2}\left[\frac{(4R + r)^2}{s^2} - 2\right](s^2 - 2r^2 - 8Rr) - 1 \overset{(1)}{\le}$$

$$\overset{(1)}{\le} \frac{1}{r^2}\left[\frac{(4R + r)^2}{\frac{r(4R + r)^2}{R + r}} - 2\right](4R^2 + 4Rr + 3r^2 - 2r^2 - 8Rr) =$$

$$= \frac{1}{r^2}\left(\frac{R}{r} - 1\right)(4R^2 - 4Rr + r^2) - 1 = \frac{1}{r^2}\left(\frac{R}{r}\right)(2R - r)^2 - 1 =$$

$$= \left(\frac{R}{r} - 1\right)\left(\frac{2R}{r} - 1\right)^2 - 1 = RHS$$ **where, (1) it follows from**

$$\frac{r(4R + r)^2}{R + r} \le 16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2 (Gerretsen)$$

Equality holds if and only if triangle is equilateral.

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