R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

ABOUT AN INEQUALITY BY ABDUL HANNAN-I

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

In ΔABC the following relationship holds:

$$\sum \frac{1}{a^2 r_a} \geq \frac{1}{2R} \sum \frac{1}{r_a^2}$$

Proposed by Abdul Hannan-Tezpur-India

Solution by Marin Chirciu-Romania

Lemma. In ΔABC the following relationship holds:

$$\sum \frac{1}{a^2 r_a} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16s^2r^3R^2}$$

Proof. Using $r_a = \frac{F}{s-a}$, we get:

$$\sum \frac{1}{a^2 r_a} = \sum \frac{1}{a^2 \frac{F}{s-a}} = \frac{1}{F}\sum \frac{s-a}{a^2} = \frac{1}{sr}\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sr^2R^2} =$$

$$= \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16s^2r^3R^2}$$

Which follows from:

$$\sum \frac{s-a}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16sr^2R^2}$$

Let's get back to the main problem.

Using Lemma and identity: $\sum \frac{1}{r_a^2} = \frac{s^2-2r(4R+r)}{s^2r^2}$, inequality can be written as:

$$\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16s^2r^3R^2} \geq \frac{1}{2R}\frac{s^2-2r(4R+r)}{s^2r^2} \Leftrightarrow$$

$$s^2(s^2+2r^2-12Rr)+r^3(4R+r) \geq 8Rr(s^2-2r^2-8Rr) \Leftrightarrow$$

$$s^2(s^2+2r^2-20Rr)+r^3(4R+r)+16Rr^2(r+4R) \geq 0 \Leftrightarrow$$

$s^2(s^2+2r^2-20Rr)+r^2(4R+r)(16R+r) \geq 0$, which follows from:

Lemma. In ΔABC the following relationship holds:

$$s^4 \geq 2s^2r(10R-r) - r^2(4R+r)(16R+r)$$

Equality holds if and only if triangle is equilateral.

Remark. Let's find an reverse inequality.

In ΔABC the following relationship holds:

$$\sum \frac{1}{a^2 r_a} \le \frac{R}{8r^2} \sum \frac{1}{r_a^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Lemma, inequality can be written as:

$$\sum \frac{1}{a^2 r_a} \le \frac{R}{8r^2} \sum \frac{1}{r_a^2} \Leftrightarrow$$

$$\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16s^2r^3R^2} \le \frac{R}{8r^2} \frac{s^2-2r(4R+r)}{s^2r^2} \Leftrightarrow$$

$$s^2r(s^2+2r^2-12Rr)+r^4(4R+r) \le 2R^3(s^2-2r^2-8Rr) \Leftrightarrow$$

$$s^2(rs^2+2r^3-12Rr^2-2R^3)+r(4R+r)(4R^3+r^3) \le 0 \Leftrightarrow$$

$$r(4R+r)(4R^3+r^3) \le s^2(2R^3+12Rr^2-2r^3-rs^2), \text{ which follows from}$$

$$16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2 (Gerretsen).$$

Remains to prove that:

$$r(4R+r)(4R^3+r^3) \le (16Rr-5r^2)\left(2R^3+12Rr^2-2r^3-r(4R^2+4Rr+3r^2)\right) \Leftrightarrow$$

$$(4R+r)(4R^3+r^3) \le (16R-5r)(2R^3-4R^2r+8Rr^2-5r^3) \Leftrightarrow$$

$$8R^4-39R^3r+74R^2r^2-62Rr^3+12r^4 \ge 0 \Leftrightarrow$$

$$(R-2r)(8R^3-23R^2r+28Rr^2-6Rr^2-6r^3) \ge 0,$$

which is true from $R \ge 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

In ΔABC the following relationship holds:

$$\frac{1}{2R} \sum \frac{1}{r_a^2} \le \sum \frac{1}{a^2 r_a} \le \frac{R}{8r^2} \sum \frac{1}{r_a^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

See up these inequalities.

Equality holds if and only of triangle is equilateral.

Remark. Let's replacing r_a with h_a.

In ΔABC the following relationship holds:

$$\frac{1}{2R}\sum\frac{1}{h_a^2}\le\sum\frac{1}{a^2h_a}\le\frac{R}{8r^2}\sum\frac{1}{h_a^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using $h_a=\frac{2F}{a}$, we get:

$$\sum\frac{1}{h_a^2}=\sum\frac{1}{a^2\frac{2F}{a}}=\frac{1}{2F}\sum\frac{1}{a}=\frac{1}{2sr}\frac{ab+bc+ca}{abc}=\frac{1}{2sr}\frac{s^2+r^2+4Rr}{4Rrs}$$

$$=\frac{s^2+r^2+4Rr}{8Rr^2s^2}$$

Let's get back to the main problem.

For RHS, using Lemma and $\sum\frac{1}{h_a^2}=\frac{s^2-r(4R+r)}{2s^2r^2}$, we have:

$$\frac{s^2+r^2+4Rr}{8Rr^2s^2}\ge\frac{1}{2R}\frac{s^2-r^2-4Rr}{2s^2r^2}\Leftrightarrow R(s^2+r^2+4Rr)\ge 4r(s^2-r^2-4Rr)$$

Which follows from $16Rr-5r^2<s^2<4R^2+4Rr+3R^2(Gerretsen)$.

Remains to prove that:

$$R(1rRr-5r^2+r^2+4Rr)\ge 4r(4R^2+4Rr+3r^2-r^2-4Rr)\Leftrightarrow$$

$$R^2-Rr-2r^2\ge 0\Leftrightarrow(R-2r)(R+r)\ge 0 \text{ true from } R\ge 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

For LHS, using Lemma and $\sum\frac{1}{h_a^2}=\frac{s^2-r(4R+r)}{2s^2r^2}$, we have:

$$\frac{s^2+r^2+4Rr}{8Rr^2s^2}\ge\frac{r}{R^2}\frac{s^2-r^2-4Rr}{2s^2r^2}\Leftrightarrow$$

$$R(s^2+r^2+4Rr)\ge 4r(s^2-r^2-4Rr) \text{ true from}$$

$$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2(Gerretsen).$$

Remains to prove that:

$$R(1rRr-5r^2+r^2+4Rr)\geq 4r(4R^2+4Rr+3r^2-r^2-4Rr)\Leftrightarrow$$

$$R^2-Rr-2r^2\geq 0\Leftrightarrow (R-2r)(R+r)\geq 0 \text{ true from } R\geq 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

In ΔABC the following relationship holds:

$$\sum\frac{1}{a^2h_a}\leq\sum\frac{1}{a^2r_a}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using up these Lemmas, we have:

$$\sum\frac{1}{a^2r_a}=\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16s^2r^3R^2}$$

$$\sum\frac{1}{a^2h_a}=\frac{s^2+r^2+4Rr}{8Rr^2s^2}$$

Inequality can be written as:

$$\frac{s^2+r^2+4Rr}{8Rr^2s^2}\leq\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16s^2r^3R^2}\Leftrightarrow$$

$s^2(s^2+2r^2-14Rr)\geq r^2(8R^2-2Rr-r^2)$, which follows from

$$s^2\geq 16Rr-5r^2(Gerretsen).$$

Remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2+2r^2-14Rr)\geq r^2(8R^2-2Rr-r^2)\Leftrightarrow$$

$$3R^2-7Rr+2r^2\geq 0\Leftrightarrow (R-2r)(3R-r)\geq 0, \text{ true from } R\geq 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro