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ABOUT A RMM INEQUALITY-V

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1) If $a,b>0$ then:

$$\sqrt[3]{\frac{a^3+b^3}{2}}\sqrt[4]{\frac{a^4+b^4}{2}}\sqrt[5]{\frac{a^5+b^5}{2}}\le\frac{a^5+b^5}{a^2+b^2}$$

Proposed by Daniel Sitaru-Romania

Solution by Marin Chirciu-Romania

Using power means inequality:

If $x_1,x_2,\dots,x_n>0,r\ge s>0$ then:

$$\left(\frac{x_1^r+x_2^r+\dots+x_n^r}{n}\right)^{\frac{1}{r}}\ge\left(\frac{x_1^s+x_2^s+\dots+x_n^s}{n}\right)^{\frac{1}{s}}\ge\sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\dots+x_n^r}{n}}\ge\sqrt[s]{\frac{x_1^s+x_2^s+\dots+x_n^s}{n}}\ge\sqrt[n]{x_1x_2\dots x_n},;r,s\in\mathbb{N},r\ge s\ge 2$$

We get: $\sqrt[3]{\frac{a^3+b^3}{2}}\le\sqrt[4]{\frac{a^4+b^4}{2}}\le\sqrt[5]{\frac{a^5+b^5}{2}}$, thus

$$LHS=\sqrt[3]{\frac{a^3+b^3}{2}}\sqrt[4]{\frac{a^4+b^4}{2}}\sqrt[5]{\frac{a^5+b^5}{2}}\le\sqrt[5]{\frac{a^5+b^5}{2}}\sqrt[5]{\frac{a^5+b^5}{2}}\sqrt[5]{\frac{a^5+b^5}{2}}=$$

$$\le\left(\sqrt[5]{\frac{a^5+b^5}{2}}\right)^3\overset{(1)}{\le}\frac{a^5+b^5}{a^2+b^2}=RHD,$$

Where $\left(\sqrt[5]{\frac{a^5+b^5}{2}}\right)^3\le\frac{a^5+b^5}{a^2+b^2}\Leftrightarrow\left(\frac{a^5+b^5}{2}\right)^3\le\left(\frac{a^5+b^5}{a^2+b^2}\right)^5\Leftrightarrow$

$(a^2+b^2)^5\le 8(a^5+b^5)^2$ which is true from Holder inequality:

$$8(a^5+b^5)^2=(1+1)(1+1)(1+1)(a^5+b^5)(a^5+b^5)\ge(a^2+b^2)^5$$

Equality if and only if $a=b$.

Remark. Inequality can be develop.

2) If $a,b>0, n\in\mathbb{N}, n\geq 2$ then:

$$\sqrt[n]{\frac{a^n+b^n}{2}}\sqrt[n+1]{\frac{a^{n+1}+b^{n+1}}{2}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\leq\frac{a^{n+2}+b^{n+2}}{a^{n-1}+b^{n-1}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using power means inequality:

If $x_1,x_2,\dots,x_n>0, r\geq s>0$ then:

$$\left(\frac{x_1^r+x_2^r+\dots+x_n^r}{n}\right)^{\frac{1}{r}}\geq\left(\frac{x_1^s+x_2^s+\dots+x_n^s}{n}\right)^{\frac{1}{s}}\geq\sqrt[n]{x_1x_2\dots x_n}$$

$$\sqrt[r]{\frac{x_1^r+x_2^r+\dots+x_n^r}{n}}\geq\sqrt[s]{\frac{x_1^s+x_2^s+\dots+x_n^s}{n}}\geq\sqrt[n]{x_1x_2\dots x_n}, ; r,s\in\mathbb{N}, r\geq s\geq 2$$

We get: $\sqrt[n]{\frac{a^n+b^n}{2}}\leq\sqrt[n+1]{\frac{a^{n+1}+b^{n+1}}{2}}\leq\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}$, **thus**

$$LHS=\sqrt[n]{\frac{a^n+b^n}{2}}\sqrt[n+1]{\frac{a^{n+1}+b^{n+1}}{2}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\leq$$

$$\leq\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}=$$

$$\leq\left(\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\right)^3\overset{(1)}{\leq}\frac{a^{n+2}+b^{n+2}}{a^{n-1}+b^{n-1}}=RHD,$$

Where $\left(\sqrt[n+2]{\frac{a^{n+2}+b^{n+2}}{2}}\right)^3\leq\frac{a^{n+2}+b^{n+2}}{a^{n-1}+b^{n-1}}\Leftrightarrow\left(\frac{a^{n+2}+b^{n+2}}{2}\right)^3\leq\left(\frac{a^{n+2}+b^{n+2}}{a^{n-1}+b^{n-1}}\right)^5\Leftrightarrow$

$(a^{n-1}+b^{n-1})^{n+2}\leq 8(a^{n+2}+b^{n+2})^{n-1}$ **which is true from Holder inequality:**

$$8(a^{n+2}+b^{n+2})^{n-1}=(1+1)(1+1)(1+1)(a^{n+2}+b^{n+2})(a^{n+2}+b^{n+2})\geq$$

$$\geq\left(\sqrt[n+2]{a^{(n+2)(n-1)}}+\sqrt[n+2]{b^{(n+2)(n-1)}}\right)^{n+2}\geq(a^{n-1}+b^{n-1})^{n+1}$$

Equality if and only if $a = b$.

Note. For $n = 2$ it follows Cyclic Inequality-730 proposed by Daniel Sitaru-R.M.M 4/2020.

Remark. Inequality can be develop for three variables.

3) If $a, b, c > 0, n \in \mathbb{N}, n \geq 2$ then:

$$\sqrt[n]{\frac{a^n + b^n + c^n}{3}} \sqrt[n+1]{\frac{a^{n+1} + b^{n+1} + c^{n+1}}{3}} \sqrt[n+2]{\frac{a^{n+2} + b^{n+2} + c^{n+2}}{3}} \leq \frac{a^{n+2} + b^{n+2} + c^{n+2}}{a^{n-1} + b^{n-1} + c^{n-1}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using power means inequality:

If $x_1, x_2, \dots, x_n > 0, r \geq s > 0$ then:

$$\left(\frac{x_1^r + x_2^r + \dots + x_n^r}{n}\right)^{\frac{1}{r}} \geq \left(\frac{x_1^s + x_2^s + \dots + x_n^s}{n}\right)^{\frac{1}{s}} \geq \sqrt[n]{x_1 x_2 \dots x_n}$$

$$\sqrt[r]{\frac{x_1^r + x_2^r + \dots + x_n^r}{n}} \geq \sqrt[s]{\frac{x_1^s + x_2^s + \dots + x_n^s}{n}} \geq \sqrt[n]{x_1 x_2 \dots x_n}, ; r, s \in \mathbb{N}, r \geq s \geq 2$$

We get: $\sqrt[n]{\frac{a^n+b^n+c^n}{3}} \leq \sqrt[n+1]{\frac{a^{n+1}+b^{n+1}+c^{n+1}}{3}} \leq \sqrt[n+2]{\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}}$

$$LHS = \sqrt[n]{\frac{a^n + b^n + c^n}{3}} \sqrt[n+1]{\frac{a^{n+1} + b^{n+1} + c^{n+1}}{3}} \sqrt[n+2]{\frac{a^{n+2} + b^{n+2} + c^{n+2}}{3}} \leq$$

$$\leq \sqrt[n+2]{\frac{a^{n+2} + b^{n+2} + c^{n+2}}{3}} \sqrt[n+2]{\frac{a^{n+2} + b^{n+2} + c^{n+2}}{3}} \sqrt[n+2]{\frac{a^{n+2} + b^{n+2} + c^{n+2}}{3}} =$$

$$= \left(\sqrt[n+2]{\frac{a^{n+2} + b^{n+2} + c^{n+2}}{3}}\right)^3 \overset{(1)}{\leq} \frac{a^{n+2} + b^{n+2} + c^{n+2}}{a^{n-1} + b^{n-1} + c^{n-1}} = RHD$$

$$(1) \Leftrightarrow \left(\sqrt[n+2]{\frac{a^{n+2} + b^{n+2} + c^{n+2}}{3}}\right)^3 \leq \frac{a^{n+2} + b^{n+2} + c^{n+2}}{a^{n-1} + b^{n-1} + c^{n-1}}$$

$$\Leftrightarrow\left(\frac{a^{n+2}+b^{n+2}+c^{n+2}}{3}\right)^{3}\le\left(\frac{a^{n+2}+b^{n+2}+c^{n+2}}{a^{n-1}+b^{n-1}+c^{n-1}}\right)^{n+2}$$

$$\Leftrightarrow(a^{n-1}+b^{n-1}+c^{n-1})^{n+2}\le 27(a^{n+2}+b^{n+2}+c^{n+2})^{n-1}$$

Which follows from Holder inequality:

$$27(a^{n+2}+b^{n+2}+c^{n+2})^{n-1}=$$

$$=(1+1+1)(1+1+1)(1+1+1)(a^{n+2}+b^{n+2}+c^{n+2})\dots(a^{n+2}+b^{n+2}+c^{n+2})\ge$$

$$\ge\left(\sqrt[n+2]{a^{(n+2)(n-1)}}+\sqrt[n+2]{b^{(n+2)(n-1)}}+\sqrt[n+2]{c^{(n+2)(n-1)}}\right)^{n+2}$$

$$=(a^{n-1}+b^{n-1}+c^{n-1})^{n+2}$$

Equality holds if and only if $a=b=c$.

REFERENCE:

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