Squared Dilogarithm and Polylogarithm Integrals

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Abstract

In these paper I will be presenting proofs for Squared Dilogarithm and Polylogarithm Integrals which is of the form $\int_0^1 \operatorname{Li}_2^2(x)dx$, $\int_0^1 \operatorname{Li}_\mu^2(x)dx$

1 Introduction

1.1 Dilogarithm Series

Dilogarithm series[3] are a special kind of infinite series which can be defined by the sum

$$\operatorname{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}; \quad |z| < 1 \tag{1}$$

1.2 Polylogarithm Series

Polylogarithms[2] are the generalised version of Dilogarithm like series and it is defined by the sum

$$\operatorname{Li}_\mu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\mu}; \quad |z| < 1 \tag{2}$$

as you can see the above series has a variable μ which can take integer values. So as previously mentioned in the case of Dilogarithm Series the value is $\mu = 2$.

1.3 Notations

Here we will be denoting a Squared Dilogarithm as

$$(\text{ Li }_2(z))^2 = \text{ Li }_2^2(x)$$

Similarly a Squared Polylogarithm is also denoted as

$$(\text{ Li }_\mu(z))^2 = \text{ Li }_\mu^2(x)$$

2 Evaluation of Integral $\int_0^1$ Li $_2^2(x)\,dx$

$$\int_0^1 \text{ Li }_2^2(x)\,dx = \sum_{k=1}^{\infty} \frac{1}{n^2} \underbrace{\int_0^1 x^n \text{ Li }_2(x)\,dx} \tag{3}$$

Here

$$\int_0^1 x^n \text{ Li }_2(x)\,dx = \frac{\zeta(2)}{(n+1)} - \frac{H_{n+1}}{(n+1)^2}$$

(The proof for the above integral will be given after this section).
On substituting the value of integral in (3) we will get

$$\begin{aligned}\int_0^1 \text{ Li }_2^2(x)\,dx &= \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{\zeta(2)}{(n+1)} - \frac{H_{n+1}}{(n+1)^2}\right); \\ &= \zeta^2(2) - \zeta(2) - \sum_{n=1}^{\infty} \underbrace{\left(\frac{H_{n+1}}{(n(n+1))^2}\right)} \\ &= \frac{\pi^4}{36} - \frac{\pi^2}{3} - 4\zeta(3) + 6\end{aligned}$$

(The underlined Harmonic sum[1] can be solved by partial sums and , I will be mentioning its proof after this section)
Therefore we get

$$\int_0^1 \text{ Li }_2^2(x)\,dx = \frac{\pi^4}{36} - \frac{\pi^2}{3} - 4\zeta(3) + 6 \tag{4}$$

2.1 Integral $\int_0^1 x^n \ \mathbf{Li}_{\,2}(x)\ dx$

$$\begin{aligned}
\int_0^1 x^n \ \text{Li}_{\,2}(x)\,dx &= \int_0^1 x^n \sum_{k=1}^{\infty} \frac{x^k}{k^2}\,dx \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2(n+k+1)} \qquad \text{(apply partial sums)} \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{(n+1)\,k^2} + \frac{1}{(n+1)^2} \left(\frac{1}{(n+k+1)} - \frac{1}{(k)} \right) \right) \\
&= \frac{\zeta(2)}{(n+1)} + \frac{1}{(n+1)^2} \sum_{k=1}^{\infty} \left(\frac{1}{(n+k+1)} - \frac{1}{(k)} \right) \\
&= \frac{\zeta(2)}{(n+1)} - \frac{H_{n+1}}{(n+1)^2} \qquad \left(\because \sum_{k=1}^{\infty} \left(\frac{1}{(k)} - \frac{1}{(n+k+1)} \right) = H_{n+1} \right)
\end{aligned}$$

Therefore

$$\int_0^1 x^n \ \text{Li}_{\,2}(x)\,dx = \frac{\zeta(2)}{(n+1)} - \frac{H_{n+1}}{(n+1)^2}; \qquad \forall n \in \mathbb{N}$$

2.2 Harmonic Series $\sum_{n=1}^{\infty} \frac{H_{n+1}}{(n\,(n+1))^2}$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_{n+1}}{(n\,(n+1))^2} &= \sum_{n=1}^{\infty} \frac{H_n}{(n\,(n+1))^2} + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^3} \\
&= -2\sum_{n=1}^{\infty} \frac{H_n}{(n\,(n+1))} + \sum_{n=1}^{\infty} \frac{H_n}{(n)^2} + \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^2} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^3}
\end{aligned}$$

let us evaluate each sum term by term

$$\sum_{n=1}^{\infty} \frac{H_n}{(n\,(n+1))} = \zeta(2)$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta\,(3)$$

$$\sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^2} = 2\zeta(3) - 1$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^3} = \zeta(3)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 (n+1)^3} = \zeta\,(3) - 6 + \frac{\pi^2}{2} \quad \text{(by partial sums)}$$

On substituting the values we will get

$$\sum_{n=1}^{\infty} \frac{H_{n+1}}{(n\,(n+1))^2} = 4\zeta\,(3) - 6 + \frac{\pi^2}{6} \tag{5}$$

3 Evaluation of the Integral $\int_0^1$ Li $^2_\mu(x)\,dx$

As you can see this integral is a generalized form of the Squared Dilogarithm integral, Before proceeding to actual problem we will have to prove a theorem which will come in handy to solve the main integral.

Theorem 1.

$$\int_0^1 x^a \; Li_{\;\mu}\,(x)\,dx = \sum_{n=1}^{\mu-1} \frac{(-1)^{n-1}\zeta\,(\mu-n+1)}{(a+1)^n} + \frac{(-1)^{\mu-1}}{(a+1)^{\mu}} H_{a+1}; \qquad \forall a \in \mathbb{N}$$

Proof. We will make use of Integration by parts

$$
\begin{aligned}
\int_0^1 x^a \text{ Li }_{\mu}(x)\,dx &= \frac{\zeta(\mu)}{(a+1)} - \frac{1}{(a+1)}\underbrace{\int_0^1 x^a \text{ Li }_{\mu-1}(x)\,dx}_{\delta(\mu-1)} \\
&= \frac{1}{(a+1)}\left(\zeta(\mu) - \delta(\mu-1)\right) \qquad \left(\because \delta(\mu-1) = \frac{1}{(a+1)}\left(\zeta(\mu-1) - \delta(\mu-2)\right)\right) \\
&= \frac{1}{(a+1)}\left(\zeta(\mu) - \frac{1}{(a+1)}\left(\zeta(\mu-1) - \delta(\mu-2)\right)\right) \\
&= \frac{1}{(a+1)}\left(\zeta(\mu) - \frac{1}{(a+1)}\left(\zeta(\mu-1) - \frac{1}{(a+1)}\left(\zeta(\mu-2) - \delta(\mu-3)\right)\right)\right) \\
&= \frac{\zeta(\mu)}{(a+1)} - \frac{\zeta(\mu-1)}{(a+1)^2} + \frac{\zeta(\mu-2)}{(a+1)^3} - .... + \frac{(-1)^{k-1}\zeta(2)}{(a+1)^k} + \frac{(-1)^{\mu-1}}{(a+1)^{\mu-1}}\delta(1) \\
&= \sum_{n=1}^{\mu-1} \frac{(-1)^{n-1}\zeta(\mu-n+1)}{(a+1)^n} + \frac{(-1)^{\mu-1}}{(a+1)^{\mu}} H_{a+1} \qquad \left(\because \delta(1) = \frac{H_{a+1}}{(a+1)}\right)
\end{aligned}
$$

□

Now let us come back to the main integral i.e $\int_0^1 \text{ Li }_{\mu}^{2}(x)\,dx$

Proof.

$$
\begin{aligned}
\int_0^1 \text{ Li }_{\mu}^{2}(x) &= \sum_{k=1}^{\infty} \frac{1}{k^{\mu}} \int_0^1 x^k \text{ Li }_{\mu}(x)dx \qquad \text{(apply theorem 1)} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^{\mu}} \sum_{n=1}^{\mu-1} \frac{(-1)^{n-1}\zeta(\mu-n+1)}{(k+1)^n} + (-1)^{\mu-1} \sum_{k=1}^{\infty} \frac{H_{k+1}}{k^{\mu}(k+1)^{\mu}}
\end{aligned}
$$

□

Therefore

$$
\int_0^1 \text{ Li }_{\mu}^{2}(x) = \sum_{k=1}^{\infty} \frac{1}{k^{\mu}} \sum_{n=1}^{\mu-1} \frac{(-1)^{n-1}\zeta(\mu-n+1)}{(k+1)^n} + (-1)^{\mu-1} \sum_{k=1}^{\infty} \frac{H_{k+1}}{k^{\mu}(k+1)^{\mu}} : \quad \forall a \in \mathbb{N} \tag{6}
$$

for $\mu = 3$ we have

$$\int_0^1 \text{Li}_3^2(x) = \sum_{k=1}^{\infty} \frac{1}{k^3} \sum_{n=1}^{2} \frac{(-1)^{n-1}\zeta(4-n)}{(k+1)^n} + \sum_{k=1}^{\infty} \frac{H_{k+1}}{k^3(k+1)^3}$$

$$\int_0^1 \text{Li}_3^2(x) = \zeta^2(3) - 10\zeta(3) - \frac{\pi^2}{3}\zeta(3) - 4\frac{\pi^2}{3} + 20 + \frac{\pi^4}{12}$$

for $\mu = 4$

$$\int_0^1 \text{Li}_4^2(x) = \frac{\pi^8}{8100} + \frac{\pi^6}{270} - \frac{\pi^4\zeta(3)}{45} + \frac{23\pi^4}{90} - 5\pi^2 - \frac{2\pi^2\zeta(3)}{3} - 30\zeta(3) + 2\zeta^2(3) - 6\zeta(5) + 70$$

4 Conclusion

In this paper we came to see on how to evaluate squared dilogarithms integrals and also came up with a generalized formula to evaluate squared polylogarithm integral, amidst of proving we also evaluated a harmonic series. At last we came up with a closed form for squared polylogarithm of Order 3 and 4.

References

[1] Eric W Weisstein. Harmonic number. *https://mathworld. wolfram. com/*, 2002.

[2] Eric W Weisstein. Polylogarithm. *https://mathworld. wolfram. com/*, 2002.

[3] Eric W Weisstein. Dilogarithm. from mathworld–a wolfram web resource, 2006.

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