Special Elliptic K Series Integrals

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Abstract

Here in this paper I will be presenting proofs for some Elliptic K series Integrals of type

$$\int_0^1 \frac{K(x)}{\sqrt{1-x^2}}dx\ ,\quad \int_0^\infty x^{a-1}K(\sqrt{-x})dx\ ,\quad \int_0^\infty x^{a-1}K(\sqrt{-x})\ln x\, dx$$

1 Introduction

The Elliptic K series [2] arises when evaluating the elliptic integral of the first kind which is defined as

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x^2\sin\theta}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 x^{2n} \tag{1}$$

Elliptic K series can also be denoted in terms of ${}_2F_1$ Hypergeometric series i.e

$$K(x) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^2\right)$$

where

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^2\right) = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(1)_n(n!)}x^{2n} \tag{2}$$

and $(a)_n = (a)(a+1)(a+2)....(a+n-1)$

2 Evaluation of the integral $\int_0^1 \frac{K(x)}{\sqrt{1-x^2}}dx$

$$\begin{aligned}
\int_0^1 \frac{K(x)}{\sqrt{1-x^2}}dx &= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 x^{2n}dx \\
&= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 x^{2n}dx \\
&= \frac{\pi}{2}\sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}dx & \text{(Apply Beta function)} \\
&= \frac{\pi}{4}\sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)} & (2n-1)!! = \frac{2^n\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}} \\
&= \frac{1}{4}\sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{2n}\Gamma^2\left(n+\frac{1}{2}\right)}{(2n)!!^2\Gamma(n+1)} & (2n)!! = 2^n n! \\
&= \frac{\sqrt{\pi}}{4}\sum_{n=0}^{\infty} \frac{\Gamma^3\left(n+\frac{1}{2}\right)}{(n!)^2\Gamma(n+1)} \\
&= \Gamma^3\left(\frac{1}{2}\right)\frac{\sqrt{\pi}}{4}\sum_{n=0}^{\infty} \frac{\Gamma^3\left(n+\frac{1}{2}\right)}{\Gamma^2(n+1)n!\Gamma^3\left(\frac{1}{2}\right)} & (a)_n = \frac{\Gamma(n+a)}{\Gamma(a)} \\
&= \Gamma^3\left(\frac{1}{2}\right)\frac{\sqrt{\pi}}{4}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^2\, n!}
\end{aligned}$$

As we can see the above series is a ${}_3F_2$ Hypergeometric series i.e

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^2\, n!} = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 1\right)$$

Here The ${}_3F_2$ Hypergeometric series can be evaluated by making use of special case of

Watson's Summation Theorem $\quad {}_3F_2(a, a, a; 1, 1; 1) = \dfrac{\Gamma\left(1-\frac{3a}{2}\right)\cos\left(\frac{\pi a}{2}\right)}{\Gamma^3\left(1-\frac{a}{2}\right)}$

In our case $a = \left(\frac{1}{2}\right)$ Therefore $\;_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;1\right) = \frac{\Gamma\left(\frac{1}{4}\right)\cos\left(\frac{\pi}{4}\right)}{\Gamma^3\left(\frac{3}{4}\right)}$

So

$$\begin{aligned}
\int_0^1 \frac{K(x)}{\sqrt{1-x^2}}dx &= \Gamma^3\left(\frac{1}{2}\right)\frac{\sqrt{\pi}}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^2\, n!} \\
&= \Gamma^3\left(\frac{1}{2}\right)\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{1}{4}\right)\cos\left(\frac{\pi}{4}\right)}{\Gamma^3\left(\frac{3}{4}\right)} \\
&= \frac{\pi^3}{4\Gamma^3\left(\frac{3}{4}\right)}
\end{aligned}$$

Therefore

$$\int_0^1 \frac{K(x)}{\sqrt{1-x^2}}dx = \frac{\pi^3}{4\Gamma^3\left(\frac{3}{4}\right)} \tag{3}$$

3 Evaluation of the Integral $\int_0^\infty x^{a-1}K(\sqrt{-x})dx$

The Integral

$$\int_0^\infty x^{a-1}K(\sqrt{-x})dx$$

can be easily evaluated using Ramanujan Master Theorem[1].

3.1 Ramanujan Master Theorem

Ramanujan Master Theorem states that if a function has an expansion of the form

$$f(x) = \sum_{n=0}^{\infty}\frac{\varphi(n)}{n!}(-x)^n$$

then the Mellin Transform of $f(x)$ is given by

$$\int_0^\infty x^{a-1}f(x)\,dx = \Gamma(a)\,\varphi(-a)$$

Here we can see that

$$K\left(\sqrt{-x}\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2(-x)^n$$

Therefore $\quad \varphi(n) = \dfrac{\pi}{2}\dfrac{((2n)!)^2}{2^{4n}(n!)^3}$. So as per Ramanujan's master Theorem

$$\begin{aligned}\int_0^\infty x^{a-1}K(\sqrt{-x})dx &= \Gamma(a)\,\varphi(-a)\\ &= \frac{\pi}{2}\Gamma(a)\frac{((-2a)!)^2}{2^{-4a}(-a!)^3}\\ &= \frac{\pi}{2}\frac{2^{4a}\Gamma(a)\,\Gamma^2(1-2a)}{\Gamma^3(1-a)} \qquad\qquad \Gamma(1-2a) = \frac{\Gamma(1-a)\,\Gamma\left(\frac{1}{2}-a\right)}{2^{2a}\sqrt{\pi}}\\ &= \frac{\Gamma(a)\,\Gamma^2\left(\frac{1}{2}-a\right)}{2\Gamma(1-a)}\end{aligned}$$

Therefore

$$\int_0^\infty x^{a-1}K(\sqrt{-x})dx = \frac{\Gamma(a)\,\Gamma^2\left(\frac{1}{2}-a\right)}{2\Gamma(1-a)} \quad \forall\, a \in \left(0, \frac{1}{2}\right) \tag{4}$$

for example if $a = \dfrac{1}{4}$

$$\int_0^\infty \frac{K(\sqrt{-x})}{\sqrt[4]{x^3}}dx = \frac{\pi^3\sqrt{2}}{\Gamma^3\left(\frac{3}{4}\right)}$$

4 Evaluation of the Integral $\int_0^\infty x^{a-1}K(\sqrt{-x})\ln x dx$

The integral

$$\int_0^\infty x^{a-1}K(\sqrt{-x})\ln x\, dx$$

is similar to the equation (4) but with an extra $\ln x$ term so for such a situation we will implement Differentiation under Integral Sign.
On Differentiating with respect to 'a' we get

$$\frac{\partial}{\partial a}\int_0^\infty x^{a-1}K(\sqrt{-x})dx = \frac{\partial}{\partial a}\left(\frac{\Gamma(a)\,\Gamma^2\left(\frac{1}{2}-a\right)}{2\Gamma(1-a)}\right)$$
$$= \frac{\beta(a)}{2\Gamma^2(1-a)}\left(\psi(a)+\psi(1-a)-2\psi\left(\frac{1}{2}-a\right)\right)$$

where $\beta(a) = \Gamma^2\left(\frac{1}{2}-a\right)\Gamma(1-a)\Gamma(a)$
and $\psi(a)$ is digamma function

$$\int_0^\infty x^{a-1}K(\sqrt{-x})\ln x dx = \frac{\beta(a)}{2\Gamma^2(1-a)}\left(\psi(a)+\psi(1-a)-2\psi\left(\frac{1}{2}-a\right)\right);\forall a\in\left(0,\frac{1}{2}\right)$$

for $a=\frac{1}{4}$ we have

$$\int_0^\infty \frac{K(\sqrt{-x})}{\sqrt[4]{x^3}}\ln x\,dx = \frac{\beta\left(\frac{1}{4}\right)}{2\Gamma^2\left(\frac{3}{4}\right)}\left(\psi\left(\frac{1}{4}\right)+\psi\left(\frac{3}{4}\right)-2\psi\left(\frac{1}{4}\right)\right)$$
$$= \frac{\pi^4\sqrt{2}}{\Gamma^3\left(\frac{3}{4}\right)}$$

for $a=\frac{1}{6}$

$$\int_0^\infty \frac{K(\sqrt{-x})}{\sqrt[6]{x^5}}\ln x\,dx = \frac{\pi\Gamma^2\left(\frac{1}{3}\right)}{\Gamma^2\left(\frac{5}{6}\right)}\left(\frac{\pi}{\sqrt{3}}-4\ln 2\right)$$

4.1 A Bonus Integral $\int_0^1 x^{a-1} K\left(\sqrt{x}\right) dx$

Here we will make use of hypergeometric series to evaluate this integral

$$\begin{aligned}
\int_0^1 x^{a-1} K\left(\sqrt{x}\right) dx &= \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k k!} \int_0^1 x^{k+a-1} dx \\
&= \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k k! \, (k+a)} \\
&= \frac{\Gamma(a+1)}{\Gamma(a)} \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 (a)_k}{(1)_k k! (a+1)_k} \\
&= \frac{\pi}{2a} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, a; 1, a+1; 1\right)
\end{aligned}$$

Therefore we get

$$\int_0^1 x^{a-1} K\left(\sqrt{x}\right) dx = \frac{\pi}{2a} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, a; 1, a+1; 1\right) \quad \forall a > 0 \tag{5}$$

for $a = \frac{1}{2}$

$$\int_0^1 \frac{K\left(\sqrt{x}\right)}{\sqrt{x}} dx = \pi {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; 1\right) = 4G \qquad \text{(G is catalan constant)}$$

for $a = \frac{3}{2}$

$$\int_0^1 \sqrt{x} K\left(\sqrt{x}\right) dx = G + \frac{1}{2}$$

5 Conclusion

In this paper I just gave some proof to evaluate some special Elliptical K series Integral's with the help of Hypergeometric Series. Elliptical Integrals and its Series originates from the study of motion of pendulum and its detailed study can be dated back to the time of Euler and Jacobi.

References

[1] D Babusci and G Dattoli. On ramanujan master theorem. *arXiv preprint arXiv:1103.3947*, 2011.

[2] Eric W Weisstein. Elliptic integral of the first kind. *https://mathworld. wolfram. com/*, 2002.

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