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ROMANIAN MATHEMATICAL MAGAZINE

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UP.342. Prove that if $0 < a \leq b$ then:

$$\left(\int_a^b \frac{\log x}{x}dx\right)^2 \geq \left(\int_{\frac{a+b}{2}}^b \frac{\log x}{x}dx + \int_{\sqrt{ab}}^b \frac{\log x}{x}dx\right)\left(\int_a^{\frac{a+b}{2}} \frac{\log x}{x}dx + \int_a^{\sqrt{ab}} \frac{\log x}{x}dx\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohammad Rostami-Kabul-Afganistan, Solution 2 by Nassim Nicholas Taleb-New York-USA,Solution 3 by proposer

Solution 1 by Mohammad Rostami-Kabul-Afganistan

$$(a-b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + 2ab + b^2 \geq 4ab \Rightarrow (a+b)^2 \geq 4ab \Rightarrow$$

$$a + b \geq 2\sqrt{ab} \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}; (I)$$

$$a \leq b \Rightarrow a + b \leq 2b \Rightarrow \frac{a+b}{2} \leq b; (II)$$

$$a \leq b \Rightarrow a^2 \leq ab \Rightarrow a \leq \sqrt{ab}; \quad (III)$$

$$\int_a^b \frac{\log x}{x}dx = \underbrace{\int_a^{\sqrt{ab}} \frac{\log x}{x}dx}_{A} + \underbrace{\int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{\log x}{x}dx}_{B} + \underbrace{\int_{\frac{a+b}{2}}^b \frac{\log x}{x}dx}_{C} = A + B + C$$

$$\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx+\int_{\sqrt{ab}}^{b}\frac{\log x}{x}dx=\underbrace{\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx}_{C}+\underbrace{\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx}_{B}+\underbrace{\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx}_{C}=C+B+C$$

$$=2C+B$$

$$\int_{a}^{\frac{a+b}{2}}\frac{\log x}{x}dx+\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx=\underbrace{\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx}_{A}+\underbrace{\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx}_{B}+\underbrace{\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx}_{A}=A+B+A$$

$$=2A+B$$

$$\left(\int_{a}^{b}\frac{\log x}{x}dx\right)^2\ge\left(\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx+\int_{\sqrt{ab}}^{b}\frac{\log x}{x}dx\right)\left(\int_{a}^{\frac{a+b}{2}}\frac{\log x}{x}dx+\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx\right)\Leftrightarrow$$

$$(A+B+C)^2\ge(2C+B)(2A+B)\Leftrightarrow$$

$$A^2+B^2+C^2+2AB+2BC+2CA\ge 4AC+2CB+2BA+B^2\Leftrightarrow$$

$$A^2+C^2+2AC\ge 4AC\ \Leftrightarrow(A-C)^2\ge 0$$

Solution 2 by Nassim Nicholas Taleb-New York-USA

$$\text{Let } G[x]=\int\frac{\log x}{x}dx$$

$$\left(G(b)-G(a)\right)^2\ge\left(\left(G(b)-G\left(\frac{a+b}{2}\right)\right)+\left(G(b)-G\left(\sqrt{ab}\right)\right)\right)\cdot$$

$$\cdot\left(\left(G\left(\frac{a+b}{2}\right)-G(a)\right)+G\left(\sqrt{ab}\right)-G(a)\right)\cdot\left(G(a)+G(b)-G\left(\frac{a+b}{2}\right)-G\left(\sqrt{ab}\right)\right)^2$$

$$\ge 0$$

Solution 3 by proposer

$$0<a\le\sqrt{ab}\le\frac{a+b}{2}\le b$$

Denote:

$$u=\int_a^{\sqrt{ab}}\frac{\log x}{x}dx;\ v=\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx;\ w=\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx$$

$$\int_{\sqrt{ab}}^{b}\frac{\log x}{x}dx=\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx+\int_{\frac{a+b}{2}}^{b}\frac{\log x}{x}dx=v+w$$

$$\int_{a}^{\frac{a+b}{2}}\frac{\log x}{x}dx=\int_{a}^{\sqrt{ab}}\frac{\log x}{x}dx+\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{\log x}{x}dx=u+v$$

Inequality to prove can be written:

$$(u+v+w)^2\geq(w+v+w)(u+v+u)$$

$$(u+v+w)^2\geq(v+2w)(v+2u)$$

$$u^2+v^2+w^2+2uv+2uw+2vw\geq v^2+2uv+2vw+4wu$$

$$u^2+w^2-2uw\geq 0\Leftrightarrow(u-w)^2\geq 0$$

Equality holds for $a=b$.

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.