R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

UP.339 Prove that for any positive real numbers a, b, c

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{1}{2}(a+b+c)\geq\frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by proposer, Solution 2 by Tran Hong-Dong Thap-Vietnam, Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand, Solution 4 by Abdul Hannan-Tezpur-India

Solution 1 by proposer

Lemma 1. For any positive real numbers a, b, c then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a^2+b^2+c^2)(a+b+c)}{ab+bc+ca}$$

Proof. The inequality is equivalent to

$$(ab+bc+ca)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\geq(a^2+b^2+c^2)(a+b+c)$$

Or

$$\frac{a^3c}{b}+\frac{b^3a}{c}+\frac{c^3b}{a}\geq a^2b+b^2c+c^2a$$

This follows from the AM-GM inequality as follows

$$\frac{a^3c}{b}+\frac{b^3a}{c}\ge 2a^2b,$$

$$\frac{b^3a}{c}+\frac{c^3b}{a}\ge 2b^2c,$$

$$\frac{c^3b}{a}+\frac{a^3c}{b}\ge 2c^2a.$$

Using the above lemma, it's enough to show that

$$\frac{(a^2+b^2+c^2)(a+b+c)}{ab+bc+ca}+\frac{1}{2}(a+b+c)\ge\frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

This is equivalent to

$$\frac{(a^2+b^2+c^2)(a+b+c)}{ab+bc+ca}-\frac{3(a^2+b^2+c^2)}{a+b+c}\ge\frac{3(a^2+b^2+c^2)}{2(a+b+c)}-\frac{1}{2}(a+b+c)$$

$$\frac{(a^2+b^2+c^2)[(a+b+c)^2-3(ab+bc+ca)]}{(ab+bc+ca)(a+b+c)}\ge\frac{3(a^2+b^2+c^2)-(a+b+c)^2}{2(a+b+c)}$$

$$\frac{(a^2+b^2+c^2)(a^2+b^2+c^2-ab-bc-ca)}{ab+bc+ca}\ge a^2+b^2+c^2-ab-bc-ca$$

$$(a^2+b^2+c^2-ab-bc-ca)\left(\frac{a^2+b^2+c^2}{ab+bc+ca}-1\right)\ge 0$$

The last inequality is clearly true and we are done.

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c)=\left(\frac{a^2}{b}-2a+b\right)+\left(\frac{b^2}{c}-2b+c\right)+\left(\frac{c^2}{a}-2c+a\right)=$$

$$=\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}$$

$$\frac{9(a^2+b^2+c^2)}{2(a+b+c)}-\frac{3(a+b+c)}{2}=\frac{9(a^2+b^2+c^2)-3(a+b+c)^2}{2(a+b+c)}=$$

$$=\frac{6(a^2+b^2+c^2-ab-bc-ca)}{2(a+b+c)}=\frac{3[(a-b)^2+(b-c)^2+(c-a)^2]}{2(a+b+c)}$$

Therefore,

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{1}{2}(a+b+c)\ge\frac{9(a^2+b^2+c^2)}{3(a+b+c)}\Leftrightarrow$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}\ge\frac{3[(a-b)^2+(b-c)^2+(c-a)^2]}{2(a+b+c)}\Leftrightarrow$$

$$\left(\frac{1}{b}-\frac{3}{2(a+b+c)}\right)(a-b)^2+\left(\frac{1}{c}-\frac{3}{2(a+b+c)}\right)(b-c)^2$$

$$+\left(\frac{1}{a}-\frac{3}{2(a+b+c)}\right)(c-a)^2\ge 0\Leftrightarrow$$

$$\frac{2(a+c)-b}{2b(a+b+c)}(a-b)^2+\frac{2(a+b)-c}{2c(a+b+c)}(b-c)^2+\frac{2(b+c)-a}{2a(a+b+c)}(c-a)^2\ge 0\Leftrightarrow$$

$$S=\sum\frac{2(a+c)-b}{2b(a+b+c)}(a-b)^2\ge 0;(1)$$

Let us denote: $S_a=\frac{2(a+c)-b}{2b(a+b+c)}, S_b=\frac{2(a+b)-c}{2c(a+b+c)}, S_c=\frac{2(b+c)-a}{2a(a+b+c)}$

If $a\ge b\ge c$ then $S_a>0, S_c>0$

$$S_a+2S_b=\frac{2(a+c)-b}{2b(a+b+c)}+2\cdot\frac{2(b+c)-a}{2a(a+b+c)}>0\Leftrightarrow$$

$$a(2a+2b-c)+2c(2b+2c-a)\ge 0\Leftrightarrow$$

$$2a^2+2ab+2bc+4c^2-3ac\ge 0 \text{ true by } 2a^2+2c^2=2(a^2+c^2)\ge 4ac>3ac\Rightarrow$$

$$2a^2+2ab+2bc+4c^2>2a^2+2c^2>3ac$$

Similarly: $S_c+2S_b>0\overset{S.O.S.}{\Longrightarrow}S\ge 0;(2)$

If $a\le b\le c$ then $S_b>0$

$$S_a+S_b=\frac{2(a+c)-b}{2b(a+b+c)}+\frac{2(a+b)-c}{2c(a+b+c)}>0\Leftrightarrow$$

$$a(2a+2b-c)+c(2b+2c-a)>0\Leftrightarrow$$

$$2a^2+2c^2-2ac+2ab+2bc\ge 0 \text{true by } 2a^2+2b^2\ge 4ac>2ac\Rightarrow$$

$$2a^2+2c^2-2ac+2ab+2bc>2a^2+2c^2-2ac>0$$

Similarly: $S_b+S_c>0\overset{S.O.S.}{\Longrightarrow}S\ge 0;(3)\Rightarrow S\ge 0,\forall a,b,c>0\Rightarrow(1)$ is true.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$, we have $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{1}{2}(a+b+c)\ge\frac{9}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}$

$$(a+b+c)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{1}{2}(a+b+c)\right)\ge\frac{9}{2}(a^2+b^2+c^2)$$

$$2(a+b+c)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+(a+b+c)^2\geq 9(a^2+b^2+c^2)$$

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^2c}{b}+\frac{c^2b}{a}+\frac{b^2a}{c}+ab+bc+ca\geq 3(a^2+b^2+c^2)$$

and since

$$\left(\frac{a^3}{b}+ab\right)+\left(\frac{b^3}{c}+bc\right)+\left(\frac{c^3}{a}+ca\right)\geq 2(a^2+b^2+c^2)$$

We will show

$$\frac{a^2c}{b}+\frac{c^2b}{a}+\frac{b^2a}{c}\geq a^2+b^2+c^2$$

Iff $a^3c^2+c^3b^2+b^3a^2\geq a^3bc+b^3ca+c^3ab$

Select $a\leq b\leq c$; $a, ax, axy, x, y\geq 1$

$$a^3(axy)^2+(axy)^2+(axy)^2(ax)^2+(ax)^3a^2$$
$$\geq a^3(axaxy)+(ax)^3(aaxy)+(axy)^3(aax)$$

Iff $x^2y^2+x^5y^3+x^3\geq x^2y+x^4y+x^4y^3$

$$y^2+x^3y^3+x\geq y+x^2y+x^2y^3$$

$y(y-1)+x^2y^3(x-1)\geq x(xy-1)\geq xy(x-1)$ true.

$y(y-1)+(x-1)(x^2y^3-xy)=(x-1)xy(xy^2-1)\geq 0$ true.

Solution 4 by Abdul Hannan-Tezpur-India

Claim: $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}$; (1)

Proof: (1)$\Leftrightarrow\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)(ab+bc+ca)\geq(a^2+b^2+c^2)(a+b+c)$

$$\Leftrightarrow\sum a^3+\sum a^2c+\sum\frac{a^3c}{b}\geq\sum a^3\sum a^2b+\sum a^2c$$

$$\Leftrightarrow\sum\frac{a^3c}{b}\geq\sum a^2b\Leftrightarrow\sum a^4c^2\geq a^3b^2c$$

Which is true by $\sum x^2\geq\sum xy$, $x=a^2c, y=b^2a, z=c^2b$

So, it is enough to prove that

$$\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}+\frac{a+b+c}{2}\geq\frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

$$\Leftrightarrow (a+b+c)^2\left(\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{1}{2}\right)\geq \frac{9(a^2+b^2+c^2)}{2}$$

Let $k=\frac{a^2+b^2+c^2}{ab+bc+ca}\geq 1$. To prove:

$(k+2)\left(k+\frac{1}{2}\right)\geq \frac{9k}{2}\Leftrightarrow (k-1)^2\geq 0$ **which is true.**

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.