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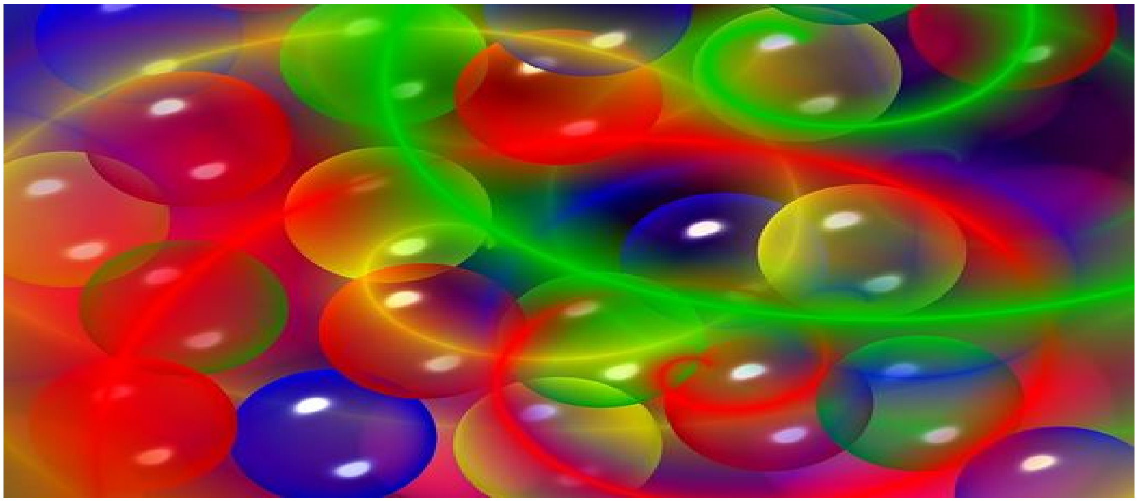

UP.332 Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ be sequences of positive real numbers such that:

$$x_1 > 1, x_{n+1} = \frac{1+(n-1)x_n^n}{nx_n^{n-1}};\ y_1 > 0, y_{n+1} = \frac{(n+1)n^n y_n}{y_n^n + n^n(n-1)}$$

Find: $\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}$

Proposed by Florică Anastase-Romania

Solution 1 by Adrian Popa-Romania, Solution 2 by proposer

Solution 1 by Adrian Popa-Romania

$$n = 1 \Rightarrow x_2 = \frac{1+0\cdot x_1^1}{1\cdot x_1^0} = 1$$

$$n = 2 \Rightarrow x_3 = \frac{1+x_2^2}{2\cdot x_2^1} = 1$$

Suppose that $x_n = 1, \forall n \in \mathbb{N}$, then $x_{n+1} = \frac{1+(n-1)\cdot 1^n}{n\cdot 1^{n-1}} = 1$.

So, $x_n = 1, \forall n \in \mathbb{N}$.

Now,

$$n = 1 \Rightarrow y_2 = \frac{2\cdot 1^1\cdot y_1}{y_1^1 + 1^1\cdot 0} = \frac{2y_1}{y_1} = 2$$

$$n=2\Rightarrow y_3=\frac{3\cdot 2^2\cdot 2}{2^2+2^2\cdot 1}=\frac{24}{8}=3$$

Suppose that $y_n=n,\forall n\in\mathbb{N}$, then we have:

$$y_{n+1}=\frac{(n+1)\cdot n^n\cdot n}{n^n+n^n(n-1)}=\frac{(n+1)n^{n+1}}{n^n(1+n-1)}=n+1.$$

So, $y_n=n,\forall n\in\mathbb{N}$.

Therefore,

$$\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}=\lim_{n\to\infty}\left(\frac{1+n}{n}\right)^{\sqrt{n}}=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{\sqrt{n}}{n}}=e^0=1$$

Solution 2 by proposer

$$x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}\overset{AM-GM}{\geq}\frac{\sqrt[n]{x_n^{n(n-1)}}}{x_n^{n-1}}=1\Rightarrow x_n\geq 1,\forall n\in\mathbb{N};\ \ (1)$$

$$x_{n+1}-x_n=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}-x_n=\frac{1+(n-1)x_n^n-nx_n^n}{nx_n^{n-1}}=\frac{1-x_n^n}{nx_n^{n-1}}\overset{(1)}{\leq}0,\forall n\in\mathbb{N}\Rightarrow$$

$(x_n)_{n\geq 1}$ –decreasing; (2) and $x_n\in(1,x_1)$.

From (1),(2) we get $(x_n)_{n\geq 1}$ –convergent; (3)

$$y_{n+1}=\frac{(n+1)n^ny_n}{y_n^n+n^n(n-1)}\Leftrightarrow\frac{y_{n+1}}{n+1}=\frac{n^ny_n}{y_n^n+n^n(n-1)}\Leftrightarrow$$

$$\frac{n+1}{y_{n+1}}=\frac{y_n^n+n^n(n-1)}{n^ny_n}=\frac{y_n^{n-1}}{n^n}+\frac{n-1}{y_n}=\frac{1}{n\left(\frac{n}{y_n}\right)^{n-1}}+\frac{n-1}{n}\cdot\frac{n}{y_n}\overset{\frac{n}{y_n}=x_n}{=}$$

$$=\frac{1}{nx_n^{n-1}}+\frac{(n-1)x_n}{n}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}=x_{n+1}$$

So,

$$\frac{n}{y_n}=x_n,\forall n\in\mathbb{N}\Rightarrow y_n=\frac{n}{x_n}\overset{(3)}{\Longrightarrow}\lim_{n\to\infty}y_n=+\infty$$

$$\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}=\lim_{n\to\infty}\left(1+\frac{x_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}=\lim_{n\to\infty}\left[\left(1+\frac{x_n}{y_n}\right)^{\frac{y_n}{x_n}}\right]^{\frac{\sqrt{n}}{y_n}}=e^{\lim\limits_{n\to\infty}\left(\frac{n}{y_n}\cdot\frac{1}{\sqrt{n}}\right)}=$$

$$=e^{\lim\limits_{n\to\infty}\frac{x_n}{\sqrt{n}}}\overset{(3)}{=}1$$