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UP.331 If $a,b,c\in(0,1), n\in\mathbb{N}, n\geq 2$ then prove:

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right)\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

Proposed by Florică Anastase-Romania

Solution 1 by Adrian Popa-Romania, Solution by proposer

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$$\sqrt[n]{sina}=\sqrt[n]{1\cdot 1\cdot\ldots\cdot 1\cdot sina}\overset{AM-GM}{\leq}\frac{1+1+\cdots+1+sina}{n}=\frac{n-1+sina}{n}\Rightarrow$$

$$1-\sqrt[n]{sina}\geq 1-\frac{n-1+sina}{n}=\frac{1-sina}{n}$$

We need to prove that:

$$\sum_{cyc}\frac{1-sina}{n}\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb};\quad (*)$$

We must to prove:

$$\frac{1-sina}{2n}+\frac{1-sinb}{2n}\geq\frac{1-sinasinb}{2n+1-sinasinb}\Leftrightarrow$$

$$1-sina+1-sinb\geq 1-sinasinb\Leftrightarrow$$

$$1-sina-sinb+sinasinb\geq 0\Leftrightarrow$$

$$(1-sina)(1-sinb)\geq 0$$

So,

$$\frac{1-sina}{2n}+\frac{1-sinb}{2n}\geq\frac{1-sinasinb}{2n+1-sinasinb};(1)$$
$$\frac{1-sinb}{2n}+\frac{1-sinc}{2n}\geq\frac{1-sinbsinc}{2n+1-sinbsinc};(2)$$
$$\frac{1-sinc}{2n}+\frac{1-sina}{2n}\geq\frac{1-sincsina}{2n+1-sincsina};(3)$$

Adding these three relations, we get:

$$\sum_{cyc}\frac{1-sina}{n}\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

Solution by proposer

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right)\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$
$$3-\sum_{cyc}\sqrt[n]{sina}\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$
$$3-\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}\geq\sum_{cyc}\sqrt[n]{sina}$$
$$\sum_{cyc}\left(1-\frac{1-sinasinb}{2n+1-sinasinb}\right)\geq\sum_{cyc}\sqrt[n]{sina}$$
$$\sum_{cyc}\frac{2n}{2n+1-sinasinb}\geq\sum_{cyc}\sqrt[n]{sina}$$
$$\sum_{cyc}\frac{1}{2n+1-sinasinb}\geq\frac{1}{2n}\sum_{cyc}\sqrt[n]{sina};\quad(1)$$

Let be the function $f:[0,1]\to\mathbb{R}$,

$$f(x)=4n-(2n+1)\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right)+\alpha x\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right),\alpha\in[0,1]$$
$$f'(x)=-\frac{2n+1}{n\sqrt[n]{x^{n-1}}}+\alpha\left(\sqrt[n]{x}+\sqrt[n]{\alpha}\right)+\frac{\alpha}{n}\sqrt[n]{x}$$
$$f''(x)=\frac{2n+1}{n^2\cdot x\sqrt[n]{x^{n-1}}}+\frac{(n+1)\alpha}{n^2\sqrt[n]{x^{n-1}}}$$
$$f''(x)>0\Rightarrow f'_{[0,1]}-increasing;f'(1)=-2+\alpha\left(\sqrt[n]{\alpha}+1\right)-\frac{1-\alpha}{n}$$

$$\alpha \in [0,1], \alpha\left(\sqrt[n]{\alpha}+1\right) \le 2 \Rightarrow f'(1) \le 0 \Rightarrow f_{[0,1]} - \text{decreasing} \Rightarrow f(x) \ge f(1), \forall x \in [0,1]$$

$$f(1) = 4n - (2n+1)\left(\sqrt[n]{\alpha}+1\right) + \alpha\left(\sqrt[n]{\alpha}+1\right) \ge 0, \forall \alpha \in [0,1]$$

Let $g(\alpha) = 4n - (2n+1)\left(\sqrt[n]{\alpha}+1\right) + \alpha\left(\sqrt[n]{\alpha}+1\right) \ge 0, \forall \alpha \in [0,1]$

$$g'(\alpha) = \frac{-(2n+1) + \sqrt[n]{\alpha^{n-1}}\left((n+1)\sqrt[n]{\alpha}+1\right)}{n\sqrt[n]{\alpha^{n-1}}} \le 0,, \forall \alpha \in [0,1] \Rightarrow g - decreasing$$

$$f(1) = g(\alpha) \ge g(1) = 0$$

We have:

$$f(\beta) = 4n - (2n+1)\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right) + \alpha\beta\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right) \ge 0 \Leftrightarrow$$

$$4n \ge (2n+1-\alpha\beta)\left(\sqrt[n]{\alpha}+\sqrt[n]{\beta}\right) \Leftrightarrow \frac{1}{2n+1-\alpha\beta} \ge \frac{\sqrt[n]{\alpha}+\sqrt[n]{\beta}}{4n}$$

Therefore,

$$\frac{1}{2n+1-sinasinb} \ge \frac{\sqrt[n]{sina}+\sqrt[n]{sinb}}{4n} \ (and\ analogs); \ (2)$$

From $(1),(2)$ ***we get***

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right) \ge \sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$