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SP.332. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

Proposed by George Apostolopoulos-Greece

Solution 1 by proposer, Solution 2 by Tran Hong-Dong Thap-Vietnam, Solution 3 by Marian Ursărescu-Romania, Solution 4 and extensions by Marin Chirciu-Romania

Solution 1 by proposer

We know that $\frac{1}{a+b}\le\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)$

So,

$$\frac{c^2}{a+b}\le\frac{1}{4}\left(\frac{c^2}{a}+\frac{c^2}{b}\right)=\frac{1}{4}\left(c\cdot\frac{c}{a}+c\cdot\frac{c}{b}\right)$$

And similarly:

$$\frac{a^2}{b+c}\le\frac{1}{4}\left(a\cdot\frac{a}{b}+a\cdot\frac{a}{c}\right) \text{ and } \frac{b^2}{c+a}\le\frac{1}{4}\left(b\cdot\frac{b}{c}+b\cdot\frac{b}{a}\right)$$

Adding up these three last inequalities, we get:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{1}{4}\left[\left(a\cdot\frac{a}{b}+a\cdot\frac{a}{c}\right)+\left(b\cdot\frac{b}{c}+b\cdot\frac{b}{a}\right)+\left(c\cdot\frac{c}{a}+c\cdot\frac{c}{b}\right)\right];\quad(*)$$

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Now, we will prove that:

$$\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}$$

Consider the substitutions $a=y+z, b=z+x, c=x+y$, where x,y,z are positive real numbers.

We know that:

$$\frac{R}{r}=\frac{abc}{4(s-a)(s-b)(s-c)}=\frac{(x+y)(y+z)(z+x)}{4xyz}$$

We have:

$$\frac{1}{(z+x)^2}+\frac{1}{(y+z)^2}\le\frac{1}{4zx}+\frac{1}{4yz}=\frac{x+y}{4xyz}$$

And multiplying by $(z+x)(y+z)$ both sides, we get:

$$\frac{y+z}{z+x}+\frac{z+x}{y+z}\le\frac{(x+y)(y+z)(z+x)}{4xyz}$$

Namely $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}$

Similarly:

$$\frac{b}{c}+\frac{c}{b}\le\frac{R}{r}\,and\,\frac{c}{a}+\frac{a}{c}\le\frac{R}{r}$$

So,

$$\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}\Leftrightarrow\frac{a^2}{b^2}+\frac{b^2}{a^2}\le\frac{R^2}{r^2}-2$$

$$\frac{b}{c}+\frac{c}{b}\le\frac{R}{r}\Leftrightarrow\frac{b^2}{c^2}+\frac{c^2}{b^2}\le\frac{R^2}{r^2}-2$$

$$\frac{c}{a}+\frac{a}{c}\le\frac{R}{r}\Leftrightarrow\frac{c^2}{a^2}+\frac{a^2}{c^2}\le\frac{R^2}{r^2}-2$$

Applying the Cauchy-Schwartz inequality, we have:

$$(a^2+b^2)\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\ge\left(a\cdot\frac{a}{b}+b\cdot\frac{b}{a}\right)^2\Rightarrow\frac{a^2}{b^2}+\frac{b^2}{a^2}\le\sqrt{a^2+b^2}\cdot\sqrt{\frac{R^2}{r^2}-2}$$

And similarly:

$$\frac{b^2}{c^2}+\frac{c^2}{b^2}\le\sqrt{b^2+c^2}\cdot\sqrt{\frac{R^2}{r^2}-2}$$

$$\frac{c^2}{a^2}+\frac{a^2}{c^2}\le\sqrt{c^2+a^2}\cdot\sqrt{\frac{R^2}{r^2}-2}$$

From the $(*)$ inequality, we get:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{1}{4}\sqrt{\frac{R^2}{r^2}-2}\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

But:

$$\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)^2\le 3(a^2+b^2+b^2+c^2+c^2+a^2)$$

$$=6(a^2+b^2+c^2)\Leftrightarrow$$

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\le\sqrt{6}\cdot\sqrt{a^2+b^2+c^2}$$

It is well-known that $a^2+b^2+c^2\le 9R^2$. So

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\le 3\sqrt{6}R$$

Namely,

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{1}{4}\sqrt{\frac{R^2-2r^2}{r^2}}\cdot 3\sqrt{6}R=\frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

Equality holds if and only if the triangle ABC is equilateral.

Solution 2 by Tran Hong-Dong Thap-Vietnam

By AM-GM inequality, we have:

$$b+c\ge 2\sqrt{bc},a+b\ge 2\sqrt{ab},c+a\ge 2\sqrt{ca}$$

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\le\frac{a^2}{2\sqrt{bc}}+\frac{b^2}{2\sqrt{ca}}+\frac{c^2}{2\sqrt{ab}}=\frac{a^2\sqrt{a}+b^2\sqrt{b}+c^2\sqrt{c}}{2\sqrt{abc}}=$$

$$=\frac{a\cdot a\sqrt{a}+b\cdot b\sqrt{b}+c\cdot c\sqrt{c}}{2\sqrt{abc}}\overset{BCS}{\le}\frac{\sqrt{(a^2+b^2+c^2)(a^3+b^3+c^3)}}{2\sqrt{abc}}=$$

$$=\frac{1}{2}\sqrt{\frac{(a^2+b^2+c^2)(a^3+b^3+c^3)}{abc}}=\frac{1}{2}\sqrt{\frac{2(s^2-4Rr-r^2)\cdot 2s(s^2-6Rr-3r^2)}{4Rrs}}=$$

$$=\frac{1}{2}\sqrt{\frac{(s^2-4Rr-r^2)(s^2-6Rr-3r^2)}{Rr}}\overset{(1)}{\leq}\frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

$$(1)\Leftrightarrow\frac{(s^2-4Rr-r^2)(s^2-6Rr-3r^2)}{Rr}\leq\frac{27R^2(R^2-2r^2)}{2r}$$

Which is true because:

$$\frac{s^2-6Rr-3r^2}{R}\leq\frac{3(R^2-2r^2)}{r};(2)$$

$$\Leftrightarrow r(s^2-6Rr-3r^2)\leq 3R(R^2-2r^2)$$

Because: $2r\leq R\Rightarrow r\leq\frac{R}{2};(3)$

$$s^2\leq 4R^2+4Rr+3r^2\Rightarrow s^2-6Rr-3r^2\leq 4R^2-2Rr\overset{(4)}{\leq}6(R^2-2r^2)$$

$(4)\Leftrightarrow 2R^2+2Rr-12r^2\geq 0\Leftrightarrow 2(R-2r)(R+3r)\geq 0$ **true by** $R\geq 2r(Euler)$.

From $(3)\&(4)\Rightarrow(2)$ **is true.**

$$s^2-4Rr-r^2<9R^2;(5)$$

$\Leftrightarrow\frac{a^2+b^2+c^2}{2}<9R^2$ **ture by:** $a^2+b^2+c^2\leq 9R^2\Rightarrow$

$$\frac{a^2+b^2+c^2}{2}<a^2+b^2+c^2<9R^2$$

From $(2)\&(5)\Rightarrow(1)$ **is true.**

Solution 3 by Marian Ursărescu-Romania

We must to show:

$$\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)^2\leq\frac{27}{8}\frac{R^2}{r^2}(R^2-2r^2);\quad(1)$$

From BCS inequality, we have:

$$\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)^2\leq(a^4+b^4+c^4)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right);\quad(2)$$

From $(1)\&(2)$ **we must show:**

$$(a^4+b^4+c^4)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\leq\frac{27}{8}\frac{R^2}{r^2}(R^2-2r^2);(3)$$

But $a^4+b^4+c^4\leq 54r^3(R-r);(4)$

$$(a+b)^2 \geq 4ab \Rightarrow \frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2} \leq \frac{1}{4}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) \leq \frac{1}{4}\frac{1}{4r^2}$$

$$=\frac{1}{16r^2}(Leuenberger);\ \ (5)$$

From $(3),(4)\&(5)$ we must show:

$$54R^3(R-r)\frac{1}{16r^2} \leq \frac{27}{8}\frac{R^2}{r^2}(R^2-2r^2) \Leftrightarrow R(R-r) \leq R^2-2r^2$$

$$\Leftrightarrow R^2-Rr \leq R^2-2r^2 \Leftrightarrow Rr \geq 2r^2 \Leftrightarrow R \geq 2r(Euler)$$

Solution 4 and extensions by Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \leq \frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

Proposed by George Apostolopoulos-Greece

Solution by Marin Chirciu

Lemma. 2) In ΔABC the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Proof. We have:

$$\sum\frac{a^2}{b+c}=\frac{\sum a^2(a+b)(a+c)}{\prod(b+c)}=\frac{4s^4(s^2-3r^2-4Rr)}{2s(s^2+r^2+2Rr)}=\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Which follows from the identities:

$$\sum a^2(a+b)(a+c)=4s^4(s^2-3r^2-4Rr)$$

$$\prod(b+c)=2s(s^2+r^2+2Rr)$$

Let's get back to the main problem.

The following inequality is much stronger by the main problem.

3) In ΔABC the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \leq \frac{Rs}{2r}$$

Using Lemma, we get:

$$\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \le \frac{Rs}{2r} \Leftrightarrow 4r(s^2-3r^2-4Rr) \le R(s^2+r^2+2Rr) \Leftrightarrow$$

$$R(s^2+r^2+2Rr) \ge 4r(s^2-3r^2-4Rr) \Leftrightarrow s^2(R-4r)+r(2R^2+17Rr+12r^2) \ge 0$$

We distinguish the cases:

Case 1) If $(R-4r) \ge 0$ is obviously true.

Case 2) If $(R-4r) < 0$, inequality it can be write:

$r(2R^2+17Rr+12r^2) \ge s^2(4r-R)$ **which is true from**

$$s^2 \le 4R^2+4Rr+3r^2 (Gerretsen)$$

It remains to prove:

$$r(2R^2+17Rr+12r^2) \ge (4R^2+4Rr+3r^2)(4r-R) \Leftrightarrow$$

$$2R^2r+17Rr^2+12r^3 \ge -4R^3-4R^2r-3Rr^2+16R^2+16Rr^2+12r^3$$

$$\Leftrightarrow 4R^3-10R^2r+4Rr^2 \ge 0 \Leftrightarrow 2R^2-5Rr+2r^2 \ge 0 \Leftrightarrow (R-2r)(2R-r) \ge 0$$

true from $R \ge 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

It is suffices to prove:

$\frac{Rs}{2r} \le \frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2} \Leftrightarrow 2s^2 \le 27(R^2-2r^2)$ **which follows from**

$$s^2 \le 4R^2+4Rr+3r^2 (Gerretsen)$$

It remains to prove:

$$2(4R^2+4Rr+3r^2) \le 27(R^2-2r^2) \Leftrightarrow 8R^2+8Rr+6r^2 \le 27R^2-54r^2$$

$\Leftrightarrow 19R^2-8Rr-60r^2 \ge 0 \Leftrightarrow (R-2r)(19R-30r) \ge 0$ **true from $R \ge 2r (Euler)$.**

Remark. The inequality can be developed.

4) In ΔABC the following relationship holds:

$$s \le \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \le \frac{Rs}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Lemma. In ΔABC the following relationship holds:

$$\sum \frac{a^2}{b+c} = \frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Proof. We have:

$$\sum \frac{a^2}{b+c} = \frac{\sum a^2(a+b)(a+c)}{\prod(b+c)} = \frac{4s^4(s^2-3r^2-4Rr)}{2s(s^2+r^2+2Rr)} = \frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Which follows from the identities:

$$\sum a^2(a+b)(a+c) = 4s^4(s^2-3r^2-4Rr)$$

$$\prod(b+c) = 2s(s^2+r^2+2Rr)$$

Let's get back to the main problem.

For RHD, using lemma, we have:

$$\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \le \frac{Rs}{2r} \Leftrightarrow 4r(s^2-3r^2-4Rr) \le R(s^2+r^2+2Rr) \Leftrightarrow$$

$$R(s^2+r^2+2Rr) \ge 4r(s^2-3r^2-4Rr) \Leftrightarrow s^2(R-4r) + r(2R^2+17Rr+12r^2) \ge 0$$

We distinguish the cases:

Case 1) If $(R-4r) \ge 0$ is obviously true.

Case 2) If $(R-4r) < 0$, inequality it can be write:

$r(2R^2+17Rr+12r^2) \ge s^2(4r-R)$ **which is true from**

$$s^2 \le 4R^2+4Rr+3r^2 (Gerretsen)$$

It remains to prove:

$$r(2R^2+17Rr+12r^2) \ge (4R^2+4Rr+3r^2)(4r-R) \Leftrightarrow$$

$$2R^2r+17Rr^2+12r^3 \ge -4R^3-4R^2r-3Rr^2+16R^2+16Rr^2+12r^3$$

$$\Leftrightarrow 4R^3-10R^2r+4Rr^2 \ge 0 \Leftrightarrow 2R^2-5Rr+2r^2 \ge 0 \Leftrightarrow (R-2r)(2R-r) \ge 0$$

true from $R \ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

For Lhs, using lemma, we have:

$$\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \ge s \Leftrightarrow 2(s^2-3r^2-4Rr) \ge s^2+r^2+2Rr \Leftrightarrow$$

$\Leftrightarrow s^2 \ge 10Rr+7r^2$ **which follows from** $s^2 \ge 16Rr-5r^2 (Gerretsen)$.

It remains to prove:

$$16Rr - 5r^2 \geq 10Rr + 7r^2 \Leftrightarrow R \geq 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.