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SP.331. ΔABC has inradius r, circumradius R, sides lengths $a = BC$, $b = AC$, $c = AB$, and altitudes h_a, h_b, h_c from the vertices A, B, C, respectively. Prove that:

$$\frac{9r^2}{R} \le \frac{c}{b+c} \cdot h_a + \frac{a}{c+a} \cdot h_b + \frac{b}{a+b} \cdot h_c \le \frac{9R}{4}$$

Proposed by George Apostolopoulos-Greece

Solution 1 by proposer, Solution 2 by Marian Ursărescu-Romania, Solution 3 by Alex Szoros-Romania, Solution 4 by Agayev Seddredin-Azerbaijan

Solution 1 by proposer

Since $bc = 2R \cdot h_a$, $ca = 2R \cdot h_b$, $ab = 2R \cdot h_c$, the right inequality becomes

$$2\left(\frac{ab^2}{a+b} + \frac{bc^2}{b+c} + \frac{ca^2}{c+a}\right) \le 9R^2$$

We know that in any triangle we have $a^2 + b^2 + c^2 \le 9R^2$

So it suffices to prove that:

$$2\left(\frac{ab^2}{a+b} + \frac{bc^2}{b+c} + \frac{ca^2}{c+a}\right) \le a^2 + b^2 + c^2 \Leftrightarrow$$

$$\left(2a^2 - \frac{2ab^2}{a+b}\right) + \left(2b^2 - \frac{2bc^2}{b+c}\right) + \left(2c^2 - \frac{2ca^2}{c+a}\right) \ge a^2 + b^2 + c^2 \Leftrightarrow$$

$$\frac{2a^3}{c+a}+\frac{2b^3}{a+b}+\frac{2c^3}{b+c}\geq a^2+b^2+c^2$$

Using Cauchy-Schwartz inequality, we have:

$$\frac{2a^3}{c+a}+\frac{2b^3}{a+b}+\frac{2c^3}{b+c}=2\left(\frac{a^4}{ca+a^2}+\frac{b^4}{ab+b^2}+\frac{c^4}{bc+c^2}\right)\geq$$

$$\geq\frac{2(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)+(ab+bc+ca)}\geq\frac{2(a^2+b^2+c^2)^2}{(a^2+b^2+c^2)+(a^2+b^2+c^2)}=a^2+b^2+c^2$$

Namely,

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c\leq\frac{9}{4}R$$

For the left inequality, we have (AM-GM):

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c\geq 3\sqrt[3]{\frac{(abc)(h_ah_bh_c)}{(a+b)(b+c)(c+a)}}$$

We know that $abc=4Rrs, R>-2r\ (Euler), s=\frac{a+b+c}{2}\geq 3\sqrt{3}r$.

Also, we know that:

$$\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)^3\geq\frac{27}{h_ah_bh_c};\ \frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$$

So,

$$\left(\frac{1}{r}\right)^3\geq\frac{27}{h_ah_bh_c}\Leftrightarrow h_ah_bh_c\geq 27r^3.$$

Now,

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c\geq\frac{3\sqrt[3]{(4Rrs)(27r^3)}}{\sqrt[3]{(a+b)(b+c)(c+a)}}\geq$$

$$\geq\frac{3\sqrt[3]{4(2r)r(3\sqrt{3}r)27r^3}}{\frac{(a+b)+(b+c)+(c+a)}{3}}=\frac{3\sqrt[3]{8\cdot 3\sqrt{3}\cdot 27r^6}}{\frac{45}{3}}=\frac{3\cdot 2\cdot\sqrt{3}\cdot 3\cdot 3r^2}{45}=\frac{27\sqrt{3}r^2}{25}$$

We know that $2s\leq 3\sqrt{3}R$, so

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c\geq\frac{9r^2}{R}$$

Namely,

$$\frac{9r^2}{R} \le \frac{c}{b+c} \cdot h_a + \frac{a}{c+a} \cdot h_b + \frac{b}{a+b} \cdot h_c \le \frac{9R}{4}$$

Equality holds if and only if the triangle ABC is equilateral.

Solution 2 by Marian Ursărescu-Romania

For the left hand:

$$\frac{c}{b+c} \cdot h_a + \frac{a}{c+a} \cdot h_b + \frac{b}{a+b} \cdot h_c \ge 3\sqrt[3]{\frac{abc \cdot h_a h_b h_c}{(a+b)(b+c)(c+a)}}$$

We must show:

$$\sqrt[3]{\frac{abc \cdot h_a h_b h_c}{(a+b)(b+c)(c+a)}} \ge \frac{3r^2}{R}; (1)$$

But $abc = 4Rrs, h_a h_b h_c = \frac{2s^2r^2}{R}$ **and** $(a+b)(b+c)(c+a) = 2s(s^2+r^2+2Rr); (2)$

From $(1)\&(2)$ **we must show that:**

$$\sqrt[3]{\frac{4Rrs \cdot 2s^2r^2}{2R(s^2+r^2+2Rr)}} \ge \frac{3r^2}{R} \Leftrightarrow \frac{4s^2r^3}{s^2+r^2+2Rr} \ge \frac{27r^6}{R^3}$$

$$\Leftrightarrow \frac{4s^2}{s^2+r^2+2Rr} \ge \frac{27r^3}{R^3}; \quad (3)$$

$$s^2+r^2+2Rr \overset{Gerretsen}{\le} 4R^2+6Rr+4r^2 \overset{Euler}{\le} 4R^2+3R^2+R^2 = 8R^2; (4)$$

From $(3)\&(4)$ **we must show:**

$\frac{s^2}{2R^2} \ge \frac{27r^3}{R^3} \Leftrightarrow s^2R \ge 2 \cdot 27r^3$ **true because** $s^2 \ge 27r^2$ $(Mitrinovic), R \ge 2r$ $(Euler)$

$$\Rightarrow s^2R \ge 2 \cdot 27r^3$$

For the right hand, we have:

$h_a = csinB = \frac{bc}{2R}$ **and similarly, then we must show:**

$$\frac{1}{2R}\left(\frac{bc^2}{b+c} + \frac{ca^2}{c+a} + \frac{ab^2}{a+b}\right) \le \frac{9R}{4} \Leftrightarrow$$

$$\frac{2bc^2}{b+c} + \frac{2ca^2}{c+a} + \frac{2ab^2}{a+b} \le 9R^2; (5)$$

But $\frac{2bc}{b+c} \le \frac{b+c}{2}$ **and similarly;** (6)

From $(5)\&(6)$ we must show:

$$\frac{(b+c)c}{2}+\frac{(a+c)a}{2}+\frac{(a+b)b}{2}\le 9R^2 \Leftrightarrow$$

$$\frac{a^2+b^2+c^2+ab+bc+ca}{2}\le 9R^2;\ \ (7)$$

But $ab+bc+ca\le a^2+b^2+c^2;\ \ (8)$

From $(7)\&(8)$ we get:

$a^2+b^2+c^2\le 9R^2$ which is true.

Solution 3 by Alex Szoros-Romania

$$\frac{1}{(b+c)^2}\le\frac{1}{4bc}\Rightarrow\frac{1}{a^2(b+c)^2}\le\frac{1}{4a^2bc}\Rightarrow$$

$$\sum_{cyc}\frac{1}{a^2(b+c)^2}\le\sum_{cyc}\frac{1}{4a^2bc}=\frac{1}{4abc}\sum_{cyc}\frac{1}{a}=\frac{1}{4abc}\sum_{cyc}\frac{bc}{abc}=\frac{1}{4(abc)^2}\sum_{cyc}bc\le$$

$$\le\frac{1}{4(abc)^2}\sum_{cyc}a^2$$

$$\Rightarrow\sum_{cyc}\frac{1}{a^2(b+c)^2}\le\frac{9R^2}{4(4RrF)^2}=\frac{9}{64F^2}\Rightarrow\sum_{cyc}\frac{4F^2}{a^2(b+c)^2},=\frac{9}{16}$$

$$\sum_{cyc}\left(\frac{2F}{a}\right)^2\frac{1}{(b+c)^2}\le\frac{9}{16}\Leftrightarrow$$

$$\sum_{cyc}\left(\frac{h_a}{b+c}\right)^2\le\frac{9}{16},\forall\Delta ABC;\ \ (1)$$

$$\sum_{cyc}\left(\frac{c}{b+c}\right)h_a=\sum_{cyc}\left(\frac{h_a}{b+c}\right)c\le\sqrt{\left(\sum_{cyc}\left(\frac{h_a}{b+c}\right)\right)^2\left(\sum_{cyc}c^2\right)^2}=\frac{9R}{4};(2)$$

$$\sum_{cyc}\frac{ch_a}{b+c}\ge 3\sqrt[3]{\frac{abch_ah_bh_c}{(a+b)(b+c)(c+a)}}=\frac{3\cdot 2}{\sqrt[3]{(a+b)(b+c)(c+a)}}$$

$$\Rightarrow\sum_{cyc}\frac{ch_a}{b+c}\ge\frac{6S}{\sqrt[3]{(a+b)(b+c)(c+a)}};\ \ (3)$$

$$\frac{6S}{\sqrt[3]{(a+b)(b+c)(c+a)}} \geq \frac{9r^2}{R};\ (4)$$

$$(4) \Leftrightarrow \frac{2RS}{3r^2} \geq \sqrt[3]{(a+b)(b+c)(c+a)} \Leftrightarrow$$

$$\frac{2Rrs}{3r^2} \geq \sqrt[3]{(a+b)(b+c)(c+a)} \Leftrightarrow \left(\frac{2Rs}{3r}\right)^3 \geq (a+b)(b+c)(c+a)$$

$$\Leftrightarrow \left(\frac{2Rs}{3r}\right)^3 \geq (a+b+c)(ab+bc+ca)-abc$$

$$\Leftrightarrow \left(\frac{2Rs}{3r}\right)^3 \geq 2s(s^2+r^2+4Rr)-4Rrs \Leftrightarrow \frac{4s^2R^3}{27r^3} \geq s^2+r^2+2Rr$$

$$\Leftrightarrow \frac{4s^2R^3}{27r^3}. = \frac{4R^3(16Rr-5r^2)}{27r^3} \geq 4R^2+4Rr+3r^2+r^2+2Rr \geq s^2+r^2+2Rr$$

$$4R^3(16Rr-5r^2) \geq 27r^3(4R^2+6Rr+4r^2)$$

$$\Leftrightarrow 2R^3(16Rr-5r^2) \geq 27r^3(2R^2+3Rr+2r^2)$$

$$(R \geq 2r \Rightarrow R^2 \geq 4r^2)$$

$$\Leftrightarrow 2R(16Rr-5r^2) \geq \frac{27}{4}r(2R^2+3Rr+2r^2)$$

$$\Leftrightarrow 8R(16Rr-5r^2) \geq 27r(2R^2+3Rr+2r^2)$$

$$\Leftrightarrow 128R^2r-40Rr^2 \geq 54R^2r+81Rr^2+54r^3$$

$$\Leftrightarrow 74R^2r-121Rr^2-54r^3 \geq 0$$

$$\Leftrightarrow 74\left(\frac{R}{r}\right)^2-121\left(\frac{R}{r}\right)-54 \geq 0; t=\frac{R}{r} \geq 2 \Rightarrow$$

$$74t^2-121t-54 \geq 0 \Leftrightarrow (t-2)(74t+27) \geq 0 \text{ true.}$$

Solution 4 by Agayev Seddredin-Azerbaijan

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c = \frac{c}{b+c}\frac{2F}{a}+\frac{a}{c+b}\frac{2F}{b}+\frac{b}{c+a}\frac{2F}{c}=$$

$$=2F\left(\frac{c}{a(b+c)}+\frac{b}{b(c+a)}+\frac{c}{c(a+b)}\right) \geq$$

$$\geq 2F\cdot 3\sqrt[3]{\frac{1}{(a+b)(b+c)(c+a)}} \geq \frac{18F}{2(a+b+c)}=\frac{9F}{2s}=\frac{9sr}{2s}=\frac{9r}{2} \geq \frac{9r^2}{R}$$

$$\frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c=2F\left(\frac{c}{a(b+c)}+\frac{b}{b(c+a)}+\frac{c}{c(a+b)}\right)\le$$

$$\le 2F\left(\frac{c}{2a\sqrt{bc}}+\frac{a}{2b\sqrt{ac}}+\frac{b}{2c\sqrt{ab}}\right)=2F\left(\frac{\sqrt{bc}}{ab}+\frac{\sqrt{ac}}{bc}+\frac{\sqrt{ab}}{ca}\right)\le$$

$$\le F\left(\frac{b+c}{2ab}+\frac{a+c}{2bc}+\frac{a+b}{2ac}\right)=F\frac{a^2+b^2+c^2+ab+bc+ca}{2abc}\le$$

$$\le F\frac{18R^2}{2abc}=\frac{abc}{4R}\frac{18R^2}{2abc}=\frac{9R}{4}$$

$$(ab+bc+ca\le a^2+b^2+c^2\le 9R^2)$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.