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JP.344. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$.

Prove that:

$$(3a^5 - 3a + 2b^3 + 34)(3b^5 - 3b + 2c^3 + 34)(3c^5 - 3c + 2a^3 + 34) \geq 6^6$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by proposer, Solution 2 by Rustam Tahmazov-Baku-Azerbaijan, Solution 3 by Eldeniz Hesenov-Georgia, Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

Solution 1 by proposer

We have: $$\begin{cases} 3a^5 - 3a + 6 \geq 6a^2 \\ 2b^3 + 1 \geq 3b^2 \end{cases} \Rightarrow$$

$$3a^5 - 3a + 2b^3 + 34 \geq 3(2a^2 + b^2 + 9) = 3\left(2a^2 + b^2 + 3(ab + bc + ca)\right) \Rightarrow$$

$$3a^5 - 3a + 2b^3 + 34 \geq 3(a + b)(2a + b + 3c) =$$

$$= 3(a + b)[(a + c) + (a + c) + (b + c)] \geq$$

$$\geq 3(a + b) \cdot 3\sqrt[3]{(a + c)^2(b + c)} = 9\sqrt[3]{(a + b)^3(a + c)^2(b + c)}$$

Similarly:

$$3b^5 - 3b + 2c^3 + 34 \geq 9\sqrt[3]{(b + c)^3(b + a)^2(a + b)}$$

$$3c^5 - 3c + 2a + 34 \geq 9\sqrt[3]{(c + a)^3(b + c)^2(a + b)}$$

Hence

$$(3a^5-3a+2b^3+34)(3b^5-3b+2c^3+34)(3c^5-3c+2a+34)\ge$$

$$\ge 9^3(a+b)^2(b+c)^2(c+a)^2$$

$$(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca)\ge\frac{8}{9}\sqrt{3(ab+bc+ca)}\cdot 3=8$$

Therefore,

$$(3a^5-3a+2b^3+34)(3b^5-3b+2c^3+34)(3c^5-3c+2a+34)\ge 6^6$$

Solution 2 by Rustam Tahmazov-Baku-Azerbaijan

$$2a^5+(a^5+4)-3a\overset{AM-GM}{\ge}2a^5+5a-3a=2a^5+2a\overset{AM-GM}{\ge}4a^3$$

$$LHS\ge\prod_{cyc}(4a^3+2b^3+30)\ge 6^6$$

$$LHS=(2a^3+2+2+2a^3+2b^3+2+24)\cdot(2+2b^3+2+2c^3+2+2b^3+24)\cdot$$

$$\cdot(2+2+2c^3+2+2c^3+2a^3+24)\overset{Holder}{\ge}[2(a+b+c)+2(ab+bc+ca)+24]^3$$

$$=\left(2\sum a+3a\right)^3\ge 6^6$$

$$(a+b+c)^2\ge 3(ab+bc+ca)=9\Rightarrow a+b+c\ge 3$$

$$LHS\ge(2\cdot 3+30)=36^6=6^6$$

Solution 3 by Eldeniz Hesenov-Georgia

$$x^5-2x+4\ge x^3+2;\ \ (1)$$

$$x^5-x^3-2x+2\ge 0\Leftrightarrow(x-1)^2(x^3+2x^2+2x+2)\ge 0$$

$$LHS=\prod_{cyc}\left(2a^5+(a^5-2a+4)-a+3+30+2b^3\right)\ge$$

$$\ge\prod_{cyc}\left(2a^5+(a^3+1)+30-a+2b^3\right)\overset{AM-GM}{\ge}$$

$$\ge\prod_{cyc}\left(2a^5+2a+30+2b^3\right)\overset{AM-GM}{\ge}$$

$$\ge\prod_{cyc}(2\cdot 2a^3+2b^3+30)=\prod_{cyc}(2a^3+2b^3+2a^3+2+2+2+24)\overset{Holder}{\ge}$$

$$\ge\left(2\sum a+2\sum ab+24\right)^3\ge\left(2\sum a+30\right)^3=(3\cdot 2+30)^3=6^6$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ and $ab+bc+ca=3, a+b+c\geq 2$ consider:

$$(3a^5-3a+2b^3+34)(3b^5-3b+2c^3+34)(3c^5-3c+2a+34)\geq$$
$$\geq(2a^5+2a+2b^3+30)(2b^5+2b+2c^3+30)(2c^5+2c+2a^3+30)\geq$$
$$\geq(4a^3+2b^3+30)(4b^3+2c^3+30(4c^3+2a^3+30)=$$
$$=2^3(a^3+a^3+b^3+15)(b^3+b^3+c^3+15)(c^3+c^3+a^3+15)\geq$$
$$\geq 2^3(9(a+b+c)+9)^3\geq 2^3+18^3=6^6$$

Note by editor:
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