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JP.343 In acute ΔABC, g_a –Gergonne's cevian the following relationship holds:

$$max\{g_a^2 \cdot cosA, g_b^2 \cdot cosB, g_c^2 \cdot cosC\} \geq r^2\left(1+\frac{r}{R}\right)\left(\frac{43}{9}-\frac{8R}{9r}\right)$$

Proposed by Radu Diaconu-Romania

Solution 1 by proposer,Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by proposer

Using the following relationship:

$$g_a^2 = (s-a)^2 + 2rh_a \ (and\ analogs)$$

$$a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr$$

$h_a + h_b + h_c \geq 9r$ and $s^2 \geq 27r^2$ we have:

$$g_a^2 + g_b^2 + g_c^2 = 3s^2 - 2s(a+b+c) + a^2 + b^2 + c^2 + 2r(h_a + h_b + h_c) \Leftrightarrow$$

$$g_a^2 + g_b^2 + g_c^2 = a^2 + b^2 + c^2 - s^2 + 2r(h_a + h_b + h_c) \Leftrightarrow$$

$$g_a^2 + g_b^2 + g_c^2 = s^2 - 2r^2 - 8Rr + 2r(h_a + h_b + h_c) \Leftrightarrow$$

$$g_a^2 + g_b^2 + g_c^2 = s^2 - 2r^2 - 8Rr + 18r^2 = s^2 - 8Rr + 16r^2 \geq 43r^2 - 8Rr.$$

WLOG, suppose: $a \leq b \leq c \Rightarrow cosA \geq cosB \geq cosC$ and

$(s-a)^2 \geq (s-b)^2 \geq (s-c)^2; 2rh_a \geq 2rh_b \geq 2rh_c$ hence

$$g_a^2 \ge g_b^2 \ge g_c^2$$

Applying Chebyshev's Inequality, it follows that:

$$g_a^2 \cdot cosA + g_b^2 \cdot cosB + g_c^2 \cdot cosC \ge \frac{1}{3}(cosA + cosB + cosC)(g_a^2 + g_b^2 + g_c^2) \ge$$

$$\ge \frac{1}{3}\left(1 + \frac{r}{R}\right)(43r^2 - 8Rr) = r^2\left(1 + \frac{r}{R}\right)\left(\frac{43}{3} - \frac{8R}{3r}\right)$$

Therefore,

$$max\{g_a^2 \cdot cosA, g_b^2 \cdot cosB, g_c^2 \cdot cosC\} \ge \frac{g_a^2 \cdot cosA + g_b^2 \cdot cosB + g_c^2 \cdot cosC}{3}$$

$$\ge r^2\left(1 + \frac{r}{R}\right)\left(\frac{43}{9} - \frac{8R}{9r}\right)$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Since in ΔABC (acute): $CosA, cosB, cosC > 0 \Rightarrow$

$$max\{g_a^2 \cdot cosA, g_b^2 \cdot cosB, g_c^2 \cdot cosC\} \ge \frac{1}{3}(g_a^2 \cdot cosA + g_b^2 \cdot cosB + g_c^2 \cdot cosC) \overset{g_a \ge h_a}{\ge}$$

$$\ge \frac{1}{3}(h_a^2 cosA + h_b^2 cosB + h_c^2 cosC) = \frac{4S^2}{3}\left(\frac{cosA}{a^2} + \frac{cosB}{b^2} + \frac{cosC}{c^2}\right) =$$

$$= \frac{bc(b^2 + c^2 - a^2) + ac(a^2 + c^2 - b^2) + ab(a^2 + b^2 - c^2)}{2} =$$

$$= \frac{1}{3 \cdot 8 \cdot R^2}\left[\sum bc(b^2 + c^2) - abc\sum a\right] =$$

$$= \frac{(a + b + c)^2(ab + bc + ca) - 2(ab + bc + ca)^2 - 2(a + b + c)abc}{24R^2} =$$

$$= \frac{(s^2 + 4Rr + r^2)(2s^2 - 8Rr - 2r^2) - 16Rrs^2}{24R^2} =$$

$$= \frac{(s^2 + 4Rr + r^2)(s^2 - 4Rr - r^2) - 8Rrs^2}{12R^2} =$$

$$= \frac{s^4 - 8Rrs^2 - (4Rr + r^2)^2}{12R^2} \overset{(1)}{\ge} r^2\left(1 + \frac{r}{R}\right)\left(\frac{43}{9} - \frac{8R}{9r}\right) = \frac{r(R + r)(43r - 8R)}{9R}$$

$$(1) \Leftrightarrow 3[s^4 - 8Rrs^2 - (4Rr + r^2)^2] \ge 4Rr(R + r)(43r - 8R)$$

$$\Leftrightarrow 3s^4 - 24Rrs^2 - 3(4Rr + r^2)^2 - 4Rr(R + r)(43r - 8R) \ge 0$$

But: $s^2 \ge 16Rr - 5r^2 = 8Rr + r(8R - 5r) \overset{R \ge 2r}{\ge} 8Rr + 11r^2 > 8Rr$

$$\Rightarrow 3s^2(s^2-8Rr)-3(4Rr+r^2)^2-4Rr(R+r)(43r-8R)\geq$$

$$\geq 3(16Rr-5r^2)(8Rr-5r^2)-3(4Rr+r^2)^2-4Rr(R+r)(43r-8R)\geq 0;\ (2)$$

$$(2)\overset{t=\frac{R}{r}\geq 2}{\Longleftrightarrow} 3(16t-5)(8t-5)-3(4t+1)^2-4t(t+1)(43-8t)\geq 0$$

$$\Leftrightarrow 32t^3+196t^2-556t+72\geq 0\Leftrightarrow 4(t-2)(8t^2+65t-9)\geq 0$$

true because $t\geq 2\Rightarrow 4(t-2)\geq 0, 8t^2+65t-9=8t^2+56t+9(t-1)\overset{t\geq 2}{>}$

$>9(t-1)\geq 9>0\Rightarrow (2)\Rightarrow (1)$ is true.

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.