RMM - Triangle Marathon 1901 - 2000

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DANIEL SITARU

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1981. **In ΔABC the following relationship holds:**

$$\frac{8F}{9R^2}\le\frac{\sin 2A+\sin 2B+\sin 2C}{\sin^2 A+\sin^2 B+\sin^2 C}\le\frac{a+b+c}{r_a+r_b+r_c}$$

Proposed by Ertan Yildirim – Turkey

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\sin 2A\cdot\left(\sum_{cyc}\sin^2 A\right)^{-1}=\frac{2F}{R^2}\left(\frac{s^2-r^2-4Rr}{2R^2}\right)^{-1}=\frac{4F}{s^2-r^2-4Rr}$$

$$\frac{4F}{s^2-r^2-4Rr}\ge\frac{8F}{9R^2}\leftrightarrow$$

$$\leftrightarrow 9R^2\ge 2(s^2-r^2-4Rr)=a^2+b^2+c^2\ (LEIBNIZ)$$

$$\frac{4F}{s^2-r^2-4Rr}\le\frac{a+b+c}{r_a+r_b+r_c}\leftrightarrow\frac{4rs}{s^2-r^2-4Rr}\le\frac{2s}{4R+r}\leftrightarrow$$

$$\leftrightarrow 2r(4R+r)\le s^2-r^2-4Rr\leftrightarrow s^2\ge 3r^2+12Rr$$

$$s^2\overset{GERRETSEN}{\ge}16Rr-5r^2\ge 3r^2+12Rr\leftrightarrow$$

$$\leftrightarrow 4Rr\ge 8r^2\leftrightarrow R\ge 2r\ (EULER)$$

1982. **In any Δ ABC,**

$$\frac{1}{2}\sum\frac{n_a g_a}{r_b r_c}+\frac{r_a^2+r_b^2+r_c^2}{s^2}\le\frac{1}{2}+\frac{R}{r}$$

Proposed by Bogdan Fuștei - Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)$$
$$=an_a^2+a(s-b)(s-c)\text{ and }b^2(s-b)+c^2(s-c)$$
$$=ag_a^2+a(s-b)(s-c)$$
$$\text{and adding the above two, we get}:(b^2+c^2)(2s-b-c)$$
$$=an_a^2+ag_a^2+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)$$
$$=2(n_a^2+g_a^2)+a^2-(b-c)^2$$

$$\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)$$

$$\Rightarrow (b-c)^2+4s(s-a)+(b-c)^2=2(n_a^2+g_a^2)$$

$$\Rightarrow n_a^2+g_a^2=2s(s-a)+(b-c)^2\Rightarrow \frac{1}{2}\sum\frac{n_a g_a}{r_b r_c}\overset{A-G}{\leq}\frac{1}{4}\sum\frac{2s(s-a)+(b-c)^2}{s(s-a)}$$

$$=\frac{3}{2}+\sum\frac{(b+c)^2-4bc}{4s(s-a)}$$

$$=\frac{3}{2}+\sum\frac{(b+c)^2}{4s(s-a)}-\sum\sec^2\frac{A}{2}=\frac{3}{2}+\frac{1}{4s}\sum\frac{(s+s-a)^2}{(s-a)}-\frac{(4R+r)^2-2s^2+3s^2}{s^2}$$

$$=\frac{1}{4s}\left(\frac{s^2(4Rr+r^2)}{r^2 s}+6s+s\right)-\frac{r_a^2+r_b^2+r_c^2}{s^2}-\frac{3}{2}=2+\frac{R}{r}-\frac{r_a^2+r_b^2+r_c^2}{s^2}-\frac{3}{2}$$

$$=\frac{1}{2}+\frac{R}{r}-\frac{r_a^2+r_b^2+r_c^2}{s^2}$$

$$\Rightarrow \frac{1}{2}\sum\frac{n_a g_a}{r_b r_c}+\frac{r_a^2+r_b^2+r_c^2}{s^2}\leq\frac{1}{2}+\frac{R}{r}\ (Proved)$$

1983. In *any* $\Delta\,ABC$,

$$\sum\frac{m_a^2}{r_b r_c}+\frac{r_a^2+r_b^2+r_c^2}{r_a r_b+r_b r_c+r_c r_a}\leq\sum\frac{n_a g_a+r r_a}{w_a^2}$$

Proposed by Bogdan Fuștei - Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{bc}{w_a^2}=\sum\frac{bc(b+c)^2}{4bcs(s-a)}\overset{(i)}{=}\sum\frac{(b+c)^2}{4s(s-a)}\ \textit{and}\ \sum\frac{m_a^2}{r_b r_c}=\sum\frac{(b-c)^2+4s(s-a)}{4s(s-a)}$$

$$=3+\sum\frac{(b+c)^2-4bc}{4s(s-a)}$$

$$=3+\sum\frac{(b+c)^2}{4s(s-a)}-\sum\sec^2\frac{A}{2}\overset{\text{via (i)}}{=}3+\sum\frac{bc}{w_a^2}-\frac{(4R+r)^2+s^2}{s^2}$$

$$=\sum\frac{bc}{w_a^2}-\frac{(4R+r)^2-2s^2}{s^2}=\sum\frac{bc}{w_a^2}-\frac{r_a^2+r_b^2+r_c^2}{r_a r_b+r_b r_c+r_c r_a}$$

$$\Rightarrow \sum\frac{m_a^2}{r_b r_c}+\frac{r_a^2+r_b^2+r_c^2}{r_a r_b+r_b r_c+r_c r_a}\overset{(ii)}{=}\frac{ab}{w_c^2}+\frac{bc}{w_a^2}+\frac{ca}{w_b^2}$$

Again, Stewart's theorem $\Rightarrow \mathbf{b^2(s-c)+c^2(s-b)}$

$$= a\mathrm{n}_a^2 + a(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c}) \text{ and } \mathrm{b}^2(\mathrm{s}-\mathrm{b})+\mathrm{c}^2(\mathrm{s}-\mathrm{c})$$

$$= a g_a^2 + a(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})$$

$$\therefore a\mathrm{n}_a^2 . a g_a^2 \ge a^2\mathrm{s}^2(\mathrm{s}-a)^2$$

$$\Leftrightarrow \{\mathrm{b}^2(\mathrm{s}-\mathrm{c})+\mathrm{c}^2(\mathrm{s}-\mathrm{b})-a(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})\}\{\mathrm{b}^2(\mathrm{s}-\mathrm{b})+\mathrm{c}^2(\mathrm{s}-\mathrm{c})$$

$$-a(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})\} \overset{(a)}{\gtrsim} a^2\mathrm{s}^2(\mathrm{s}-a)^2$$

Let $\mathrm{s}-a=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ **and** $\mathrm{s}-\mathrm{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow a=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ **and** c

$$=\mathrm{x}+\mathrm{y}$$

Using these substitutions, (a)

$$\Leftrightarrow \{\mathrm{z}(\mathrm{z}+\mathrm{x})^2+\mathrm{y}(\mathrm{x}+\mathrm{y})^2-\mathrm{yz}(\mathrm{y}+\mathrm{z})\}\{\mathrm{y}(\mathrm{z}+\mathrm{x})^2+\mathrm{z}(\mathrm{x}+\mathrm{y})^2-\mathrm{yz}(\mathrm{y}+\mathrm{z})\}$$

$$\ge \mathrm{x}^2(\mathrm{y}+\mathrm{z})^2(\mathrm{x}+\mathrm{y}+\mathrm{z})^2$$

$$\Leftrightarrow \mathrm{xy}^2+\mathrm{xz}^2+\mathrm{y}^3+\mathrm{z}^3 \ge 2\mathrm{xyz}+\mathrm{yz}(\mathrm{y}+\mathrm{z}) \Leftrightarrow \mathrm{x}(\mathrm{y}-\mathrm{z})^2+(\mathrm{y}+\mathrm{z})(\mathrm{y}-\mathrm{z})^2 \ge 0 \to \text{true}$$

$$\Rightarrow (a) \text{ is true} \Rightarrow \mathrm{n}_a g_a \ge \mathrm{s}(\mathrm{s}-a)$$

$$\Rightarrow \mathrm{n}_a g_a + \mathrm{rr}_a \ge \mathrm{s}(\mathrm{s}-a)+\frac{(\mathrm{s}-a)(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})}{\mathrm{s}-a} = \mathrm{s}^2 - \mathrm{s}a + \mathrm{s}^2 - \mathrm{s}(\mathrm{b}+\mathrm{c}) + \mathrm{bc}$$

$$= 2\mathrm{s}^2 - \mathrm{s}(a+\mathrm{b}+\mathrm{c}) + \mathrm{bc} \Rightarrow \frac{\mathrm{n}_a g_a + \mathrm{rr}_a}{\mathrm{w}_a^2} \ge \frac{\mathrm{bc}}{\mathrm{w}_a^2}$$

and analogs $\Rightarrow \sum \frac{\mathrm{n}_a g_a + \mathrm{rr}_a}{\mathrm{w}_a^2}$

$$\ge \frac{a\mathrm{b}}{\mathrm{w}_\mathrm{c}^2} + \frac{\mathrm{bc}}{\mathrm{w}_a^2} + \frac{\mathrm{c}a}{\mathrm{w}_\mathrm{b}^2} \overset{\text{via (ii)}}{\cong} \sum \frac{\mathrm{m}_a^2}{\mathrm{r}_\mathrm{b}\mathrm{r}_\mathrm{c}} + \frac{\mathrm{r}_a^2+\mathrm{r}_\mathrm{b}^2+\mathrm{r}_\mathrm{c}^2}{\mathrm{r}_a\mathrm{r}_\mathrm{b}+\mathrm{r}_\mathrm{b}\mathrm{r}_\mathrm{c}+\mathrm{r}_\mathrm{c}\mathrm{r}_a} \ (\textit{Proved})$$

1984. **In** ***any*** Δ **ABC,**

$$\sum\left(\frac{\mathrm{m}_a}{\mathrm{w}_a}+\frac{\mathrm{rr}_a}{\mathrm{w}_a^2}\right) \ge 2+\frac{\mathrm{R}}{\mathrm{r}}$$

Proposed by Bogdan Fuștei - Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\left(\frac{m_a}{w_a}+\frac{rr_a}{w_a^2}\right)=\sum\left(\frac{m_aw_a}{w_a^2}+\frac{rr_a}{w_a^2}\right)\overset{\text{Ioscu and }A-G}{\geq}\sum\frac{s(s-a)+\frac{(s-a)(s-b)(s-c)}{s-a}}{w_a^2}$$

$$=\sum\frac{s^2-sa+s^2-s(b+c)+bc}{w_a^2}$$

$$=\sum\frac{2s^2-s(a+b+c)+bc}{w_a^2}=\sum\frac{bc(b+c)^2}{4bcs(s-a)}=\sum\frac{(b+c)^2}{4s(s-a)}=\frac{1}{4s}\sum\frac{(s+s-a)^2}{(s-a)}$$

$$=\frac{1}{4s}\left(\frac{s^2(4Rr+r^2)}{r^2s}+6s+s\right)=2+\frac{R}{r}\text{ QED}$$

1985. In any Δ ABC,

$$\frac{R}{r}+\lambda\frac{(b+c)^4}{b^4+c^4}\geq 8\lambda+2\ \forall\ \lambda\leq\frac{1}{12}$$

Proposed by Marin Chirciu - Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{R}{r}+\lambda\frac{(b+c)^4}{b^4+c^4}\geq 8\lambda+2\Leftrightarrow\frac{R}{r}-2\overset{(a)}{\geq}\lambda\left(8-\frac{(b+c)^4}{b^4+c^4}\right)$$

$$\frac{R}{r}-2\overset{\text{Bandila}}{\geq}\frac{b}{c}+\frac{c}{b}-2\overset{?}{\geq}\frac{1}{12}\left(8-\frac{(b+c)^4}{b^4+c^4}\right)$$

$$=\frac{7b^4+7c^4-14b^2c^2-4b^3c-4bc^3+8b^2c^2}{12(b^4+c^4)}$$

$$=\frac{7(b^2-c^2)^2-4bc(b-c)^2}{12(b^4+c^4)}$$

$$\Leftrightarrow\frac{(b-c)^2}{bc}\overset{?}{\geq}\frac{7(b^2-c^2)^2-4bc(b-c)^2}{12(b^4+c^4)}\Leftrightarrow\frac{1}{bc}\overset{?}{\geq}\frac{7(b+c)^2-4bc}{12(b^4+c^4)}\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow 12(b^4+c^4)\overset{?}{\geq}7bc(b+c)^2-4b^2c^2$$

$$\Leftrightarrow 12b^4+12c^4-7b^3c-7bc^3-10b^2c^2\underset{(i)}{\overset{?}{\geq}}0$$

$$\text{Now, } b^4 + c^4 \geq \frac{1}{2}(b^2 + c^2)^2 \overset{A-G}{\gtreqqless} bc(b^2 + c^2)$$

$$\Rightarrow 7b^4 + 7c^4 \overset{(ii)}{\gtreqqless} 7b^3c + 7bc^3 \text{ and } 5b^4 + 5c^4 - 10b^2c^2 = 5(b^2 - c^2)^2 \overset{(iii)}{\gtreqqless} 0$$

$$\therefore (ii) + (iii) \Rightarrow 12b^4 + 12c^4 - 7b^3c - 7bc^3 - 10b^2c^2 \geq 0 \Rightarrow (i) \text{ is true } \therefore \frac{R}{r} - 2$$

$$\geq \frac{1}{12}\left(8 - \frac{(b+c)^4}{b^4 + c^4}\right) \overset{\frac{1}{12} \geq \lambda}{\gtreqqless} \lambda\left(8 - \frac{(b+c)^4}{b^4 + c^4}\right)$$

$$\left(\because 8 - \frac{(b+c)^4}{b^4 + c^4} \geq 0 \text{ as } b^4 + c^4 \geq \frac{1}{8}(b+c)^4\right) \Rightarrow (a) \text{ is true } \therefore \frac{R}{r} + \lambda\frac{(b+c)^4}{b^4 + c^4}$$

$$\geq 8\lambda + 2 \ \forall \lambda \leq \frac{1}{12} \ (Proved)$$

1986. In acute ΔABC let ΔDEF be the orthic triangle, $\Delta A_1B_1C_1$ the circumcevian triangle of orthocentre in ΔABC and $\Delta A_2B_2C_2$ the circumcevian triangle of incenter in ΔDEF. Prove that:

$$\frac{[A_2B_2C_2]}{[A_1B_1C_1]} \geq \left(\frac{R}{4r}\right)^2$$

Proposed by Marian Ursărescu-Romania

Solution by Abdul Hannan-Tezpur-India

$$[A_1B_1C_1] = 8[ABC]cosAcosBcosC$$

The angles of DEF are $\pi - 2A, \pi - 2B, \pi - 2C$ and $R_{DEF} = \frac{R}{2}$

The angles of $\Delta A_1B_1C_1$ are $\frac{D+E}{e}, \frac{E+F}{2}, \frac{F+D}{2}$ or A, B, C.

$$\Rightarrow [A_2B_2C_2] = 2(R_{DEF})^2 sinAsinBsinC = \frac{2R^2 sinAsinBsinC}{4} = \frac{[ABC]}{4}$$

Also, $cosAcosBcosC = \frac{s^2 - (2R+r)^2}{4R^2}$

$$\frac{[A_2B_2C_2]}{[A_1B_1C_1]}=\frac{1}{32cosAcosBcosC}=\frac{R^2}{8(s^2-(2R+r)^2)}\overset{Gerretsen}{\geq}$$

$$\geq\frac{R^2}{8\left(4R^2+4Rr+3r^2-(4R^2+4Rr+r^2)\right)}=\frac{R^2}{16r^2}=\left(\frac{R}{4r}\right)^2$$

1987. In ΔABC the following relationship holds:

$$w_a>s_a\Leftrightarrow 1+cosA>\frac{a^2}{2(b^2+c^2)}$$

Proposed by Adil Abdullayev and Rahim Shahbazov-Baku-Azerbaijan

Solution by Abdul Hannan-Tezpur-India

$$w_a=\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}=\frac{\sqrt{bc}}{b+c}\sqrt{(b+c)^2-a^2}$$

$$s_a=\frac{2bcm_a}{b^2+c^2}=\frac{bc}{b^2+c^2}\sqrt{2b^2+2c^2-a^2}$$

$$w_a^2-s_a^2=\frac{bc((b+c)^2-a^2)}{(b+c)^2}-\frac{b^2c^2(2b^2+2c^2-a^2)}{(b^2+c^2)^2}$$

$$\Rightarrow\frac{(w_a^2-s_a^2)(b+c)^2(b^2+c^2)^2}{bc}=((b+c)^2-a^2)-bc(2b^2+2c^2-a^2)(b+c)^2=$$

$$=(b+c)^2(b^2+c^2)^2-a^2(b^2+c^2)^2-bc(2b^2+2c^2)(b+c)^2+a^2bc(b+c)^2=$$

$$=(b+c)^2(b^2+c^2)[(b^2+c^2)-2bc]-a^2[(b^2+c^2)^2-bc(b+c)^2]=$$

$$=(b+c)^2(b^2+c^2)(b-c)^2-a^2[(b^4+2b^2c^2+c^4)-(b^3c+2b^2c^2+bc^3)]=$$

$$=(b+c)^2(b^2+c^2)(b-c)^2-a^2(b-c)(b^3-c^3)=$$

$$=(b-c)^2[(b+c)^2(b^2+c^2)-a^2(b^2+bc+c^2)]$$

Dividing both sides by $2bc(b^2+c^2)$, we obtain:

$$\frac{(w_a^2-s_a^2)(b+c)^2(b^2+c^2)}{b^2c^2}=(b-c)^2\left[\frac{(b+c)^2}{2bc}-\frac{a^2}{2}\cdot\frac{b^2+bc+c^2}{bc(b^2+c^2)}\right]=$$

$$=(b-c)^2\left[1+\frac{b^2+c^2}{2bc}-\frac{a^2}{2}\cdot\left(\frac{1}{bc}+\frac{1}{b^2+c^2}\right)\right]=$$

$$= (b-c)^2\left[1+\frac{b^2+c^2-a^2}{2bc}-\frac{a^2}{2(b^2+c^2)}\right] = (b-c)^2\left[1+cosA-\frac{a^2}{2(b^2+c^2)}\right]$$

This shows that: $w_a > s_a \Leftrightarrow 1+cosA > \frac{a^2}{2(b^2+c^2)}$

1988. In ΔABC the following relationship holds:

$$\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}+\frac{2(r_a^2+r_b^2+r_c^2)}{r_ar_b+r_br_c+r_cr_a} \le 1+\frac{2R}{r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Bogdan Fuştei-Romania

Lemma: In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a^2}{r_br_c}+\frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a} = 2+\frac{R}{r}$$

$$s_a = \frac{2bc}{b^2+c^2}\cdot m_a \Rightarrow \frac{m_a}{s_a} = \frac{b^2+c^2}{2bc} = \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$w_a^2 = \frac{4bc}{(b+c)^2}\cdot r_br_c \Rightarrow \frac{r_br_c}{w_a^2} = \frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)+\frac{1}{2}$$

$$\frac{r_br_c}{w_a^2} = \frac{1}{2}+\frac{1}{2}\cdot\frac{m_a}{s_a} = \frac{1}{2}\left(1+\frac{m_a}{s_a}\right)$$

Adding, we get:

$$\sum_{cyc}\frac{r_br_c}{w_a^2} = \frac{1}{2}\left(3+\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\right)$$

But $m_aw_a \ge s(s-a) = r_br_c$

$$\frac{m_aw_a}{w_a^2} = \frac{m_a}{w_a} \ge \frac{r_br_c}{w_a^2} = \frac{1}{2}\left(1+\frac{m_a}{s_a}\right)$$

Hence,

$$\sum_{cyc}\frac{m_a}{w_a} \ge \frac{1}{2}\left(3+\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\right); \ (1)$$

$$\sum_{cyc}\frac{m_a^2}{w_a}+\frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}\le 2+\frac{R}{r};\ (2)$$

From $(1),(2)$ we get: $\frac{1}{2}\left(3+\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}\right)+\frac{r_a^2+r_b^2+r_c^2}{r_ar_b+r_br_c+r_cr_a}\le 2+\frac{R}{r}$

$$3+\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}+\frac{2(r_a^2+r_b^2+r_c^2)}{r_ar_b+r_br_c+r_cr_a}\le 2\left(2+\frac{R}{r}\right)$$

Therefore,

$$\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}+\frac{2(r_a^2+r_b^2+r_c^2)}{r_ar_b+r_br_c+r_cr_a}\le 1+\frac{2R}{r}$$

1989. In any Δ ABC,

$$1+\left(\frac{1}{c^2}-\frac{1}{(a+b)^2}\right)(a-b)^2\le\frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

Proof : $1+\left(\frac{1}{c^2}-\frac{1}{(a+b)^2}\right)(a-b)^2\le\frac{R}{2r}\Leftrightarrow\frac{R}{2r}-1\ge\left(\frac{1}{c^2}-\frac{1}{(a+b)^2}\right)(a-b)^2$

$$=\left(\frac{(a+b)^2-c^2}{c^2(a+b)^2}\right)(a-b)^2$$

$$\Leftrightarrow\frac{R}{2r}-1\overset{(a)}{\ge}\frac{4s(s-c)(a-b)^2}{c^2(a+b)^2}$$

Now, $r_a+r_b=s\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}\right)=\frac{s\sin\left(\frac{A+B}{2}\right)\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{C}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{C}{2}$

$$\therefore r_a+r_b\overset{(i)}{=}4R\cos^2\frac{C}{2}$$

$$\text{Now, } (a+b)^2 \geq 32Rr\cos^2\frac{C}{2} \overset{\text{by (i)}}{\cong} 8r(r_a+r_b) = 8r^2s\left(\frac{1}{s-a}+\frac{1}{s-b}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{c}{(s-a)(s-b)} = 4c(a+b-c)$$

$$\Leftrightarrow (a+b)^2 + 4c^2 - 4c(a+b) \geq 0 \Leftrightarrow (a+b-2c)^2 \geq 0 \to \text{true} \therefore a+b \geq 4\sqrt{2Rr}\cos\frac{C}{2}$$

$$\Rightarrow 4R\cos\frac{C}{2}\cos\frac{A-B}{2} \geq 4\sqrt{2Rr}\cos\frac{C}{2}$$

$$\Rightarrow \cos\frac{A-B}{2} \geq \sqrt{\frac{2r}{R}} \Rightarrow \frac{1}{\cos^2\frac{A-B}{2}} \overset{\text{(ii)}}{\leq} \frac{R}{2r}$$

$$\text{Now, } \frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} = \frac{\left(16R^2\sin^2\frac{A-B}{2}\sin^2\frac{C}{2}\right)\left(16R^2\cos^2\frac{C}{2}\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right)}{\left(16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\right)\left(16R^2\cos^2\frac{A-B}{2}\cos^2\frac{C}{2}\right)}$$

$$= \frac{1-\cos^2\frac{A-B}{2}}{\cos^2\frac{A-B}{2}} = \frac{1}{\cos^2\frac{A-B}{2}} - 1 \overset{\text{via (ii)}}{\leq} \frac{R}{2r} - 1$$

$$\Rightarrow \frac{R}{2r} - 1 \overset{\text{(iii)}}{\geq} \frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} \therefore \text{(iii)} \Rightarrow \text{in order to prove } (a), \text{it suffices to prove}$$

$$: \frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} \geq \frac{4s(s-c)(a-b)^2}{c^2(a+b)^2}$$

$$\Leftrightarrow \frac{(s-c)}{r^2} \geq \frac{4s}{c^2} \Leftrightarrow (s-c)c^2 \geq 4sr^2 = 4(s-a)(s-b)(s-c) \Leftrightarrow c^2$$

$$\geq (b+c-a)(c+a-b) = c^2 - (a-b)^2 \Leftrightarrow (a-b)^2 \geq 0$$

$$\to \text{true} \Rightarrow (a) \text{ is true} \therefore 1 + \left(\frac{1}{c^2} - \frac{1}{(a+b)^2}\right)(a-b)^2 \leq \frac{R}{2r}\ (Proved)$$

1990. In any $\Delta ABC, \frac{R}{2r} \geq \sqrt{1 + \frac{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}}$

Proposed by Bogdan Fuștei - Romania

Solution by Soumava Chakraborty-Kolkata-India

Proof : Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$

Now, $\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)} \overset{(a)}{=} \frac{(\sum x)^3}{xyz}$ and $1+\frac{4R}{r}=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod(y+z)}{xyz} \overset{(b)}{=} \frac{xyz+\prod(y+z)}{xyz}$

$\sum\frac{a}{b}=\sum\frac{y+z}{z+x} \overset{(c)}{=} \frac{\sum(x+y)(y+z)^2}{\prod(y+z)} \therefore (a),(b),(c) \Rightarrow \frac{s^2}{r^2} \ge \left(\sum\frac{a}{b}\right)\left(1+\frac{4R}{r}\right) \Leftrightarrow \frac{(\sum x)^3}{xyz} \ge \left[\frac{xyz+\prod(y+z)}{xyz}\right]\left[\frac{\sum(x+y)(y+z)^2}{\prod(y+z)}\right]$

$\Leftrightarrow \{\prod(y+z)\}(\sum x)^3 \ge \{xyz+\prod(y+z)\}(\sum(x+y)(y+z)^2)$

$$\Leftrightarrow \sum x^2y^4+\sum x^3y^3 \overset{(i)}{\geq} xyz(\sum x^2y)+3x^2y^2z^2$$

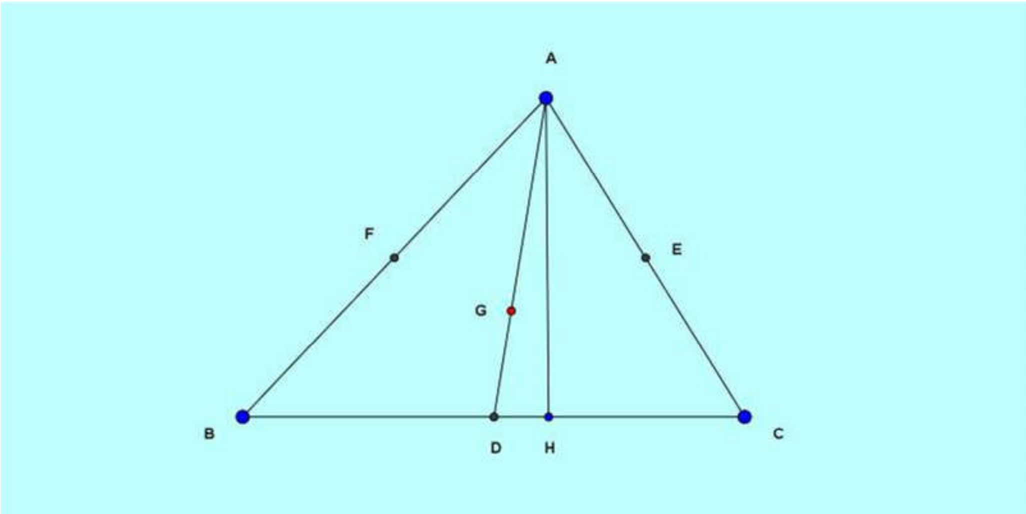
Now, if $u, v, w > 0$, then

: $v^3+v^3+u^3 \overset{A-G}{\geq} 3v^2u, w^3+w^3+v^3 \overset{A-G}{\geq} 3w^2v$ and $u^3+u^3+w^3 \overset{A-G}{\geq} 3u^2w$ and adding these three :

$\sum u^3 \ge \sum uv^2$ and choosing $u=xy, v=yz$ and $w=zx$, we get

: $\sum x^3y^3 \overset{(ii)}{\geq} xyz(\sum x^2y)$ and $\sum x^2y^4 \underset{(iii)}{\overset{A-G}{\geq}} 3x^2y^2z^2 \therefore (ii)+(iii) \Rightarrow$

(i) is true $\Rightarrow \frac{s^2}{r^2} \ge \left(\sum\frac{a}{b}\right)\left(1+\frac{4R}{r}\right) \Rightarrow \frac{s^4}{r^4} \ge \frac{(4R+r)^2}{r^2}\left(\sum\frac{a}{b}\right)^2 \Rightarrow \frac{s^2}{r^2} \ge \frac{(4R+r)^2}{s^2}\left(\sum\frac{a}{b}\right)^2 \overset{Trucht}{\geq} 3\left(\sum\frac{a}{b}\right)^2 \Rightarrow r^2 \overset{(l)}{\leq} \frac{s^2}{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2}$



$\mathbf{AD = m_a}$ $\boldsymbol{and}$ $\mathbf{AH = h_a \therefore BH = ccosB \Rightarrow DH = ccosB - \frac{\boldsymbol{a}}{2}}$ $\boldsymbol{and}$ $\mathbf{CH = bcosC \Rightarrow DH}$

$$\mathbf{= \frac{\boldsymbol{a}}{2} - bcosC \therefore 2DH = ccosB - bcosC}$$

Here c > b and proceeding in a similar manner when b > c, $\mathbf{2DH = bcosC - ccosB}$

$$\mathbf{\therefore 2DH = |bcosC - ccosB|}$$

$$\mathbf{= \left|b\left(\frac{\boldsymbol{a}^2 + b^2 - c^2}{2\boldsymbol{a}b}\right) - c\left(\frac{c^2 + \boldsymbol{a}^2 - b^2}{2c\boldsymbol{a}}\right)\right| = \frac{2\left|b^2 - c^2\right|}{2\boldsymbol{a}} \Rightarrow 4DH^2 = \frac{(b^2 - c^2)^2}{\boldsymbol{a}^2}}$$

$$\mathbf{\Rightarrow AD^2 - AH^2 = \frac{(b^2 - c^2)^2}{4\boldsymbol{a}^2} \Rightarrow m_a^2 - h_a^2 = \frac{(b^2 - c^2)^2}{4\boldsymbol{a}^2}}$$

$$\mathbf{\Rightarrow \frac{m_a^2 - h_a^2}{h_a^2} = \frac{(b^2 - c^2)^2}{4\boldsymbol{a}^2\left(\frac{4r^2s^2}{\boldsymbol{a}^2}\right)} = \frac{(b^2 - c^2)^2}{16r^2s^2} \overset{via\,(l)}{\geq} \frac{(b^2 - c^2)^2}{16\left(\frac{s^2}{3\left(\frac{\boldsymbol{a}}{b} + \frac{b}{c} + \frac{c}{\boldsymbol{a}}\right)^2}\right)s^2}}$$

$$\mathbf{= \frac{3\left(\frac{\boldsymbol{a}}{b} + \frac{b}{c} + \frac{c}{\boldsymbol{a}}\right)^2 (b^2 - c^2)^2}{(2s)^4} = \frac{3\left(\frac{\boldsymbol{a}}{b} + \frac{b}{c} + \frac{c}{\boldsymbol{a}}\right)^2 (b^2 - c^2)^2}{(\boldsymbol{a} + b + c)^4}}$$

$$\Rightarrow \frac{m_a^2}{h_a^2} \geq 1 + \frac{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 (b^2-c^2)^2}{(a+b+c)^4} \Rightarrow \sqrt{1 + \frac{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 (b^2-c^2)^2}{(a+b+c)^4}}$$

$$\leq \frac{m_a}{h_a} \overset{Panaitopol}{\leq} \frac{R}{2r} \ (Proved)$$

1991. **In ΔABC the following relationship holds:**

$$\sum_{cyc} m_a w_a \geq \frac{\prod(a-b)^2 + 4\left(\left(\prod r_a\right)^2 + r^3\left(\sum r_a\right)^3\right)}{4r^2(4R^2+20Rr-2r^2)}$$

Proposed by Soumava Chakrabory-Kolkata-India

Solution by Tran Hong-Dong Thap-Vietnam

$$\prod(a-b)^2 = \sum a^2b^2(a^2+b^2) - 2\sum a^3b^3 - 2abc(a^3+b^3+c^3) +$$

$$+2abc(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2) - 6a^2b^2c^2 = \phi$$

$$\sum a^2b^2(a^2+b^2) = \left(\sum a^2\right)\left(\sum a^2b^2\right) - 3(abc)^2 =$$

$$= 2(s^2-4Rr-r^2)[s^4+(2r^2-8Rr)s^2+(4Rr+r^2)^2] - 3\cdot 16R^2r^2s^2 =$$

$$= 2[s^6+(r^2-12Rr)s^4+(48R^2r^2+8Rr^3-r^4)s^2-(4Rr+r^2)^3] - 48R^2r^2s^2 =$$

$$= 2s^6+(2r^2-24Rr)s^4+(48R^2r^2+16Rr^3-2r^4)s^2-2(4Rr+r^2)^3$$

$$-2\sum a^3b^3 = -2s^6+(24Rr-6r^2)s^4-6r^4s^2-2(4Rr+r^2)^3$$

$$-2abc(a^3+b^3+c^3) = -8Rrs[2s^3-(12Rr+6r^2)s]$$

$$= -16Rr^4+(96R^2r^2+48Rr^3)s^2$$

$$2abc\sum ab(a+b) = 8Rrs(2s^3-4Rrs+2r^2s) = 16Rrs^4+(16Rr^3-32R^2r^2)s^2$$

$$\prod(a-b)^2 = -4r^2s^4(16R^2r^2+80Rr^3-8r^4)s^2-4(4Rr+r^2)^3 =$$

$$= r^2[-4s^4+(16R^2+80Rr-8r^2)s^2-4r(4R+r^2)^3]$$

$$\frac{\prod(a-b)^2+4\left(\left(\prod r_a\right)^2+r^3\left(\sum r_a\right)^3\right)}{4r^2(4R^2+20Rr-2r^2)}=$$

$$=\frac{r^2[-4s^4+(16R^2+80Rr-8r^2)s^2-4r(4R+r)^3+4s^4+4r(4R+r)^3]}{4r^2(4R^2+20Rr-2r^2)}=$$

$$=\frac{(16R^2+80Rr-8r^2)s^2+4r(4R+r)^3-4r(4R+r)^3}{4(4R^2+20Rr-2r^2)}=s^2;\ (1)$$

On the other hand, we known that: $m_aw_a\geq s(s-a)$ and analogs.

$$\Rightarrow\sum_{cyc}m_aw_a\geq\sum_{cyc}s(s-a)=s^2;(2)$$

From (1),(2) we get;

$$\sum_{cyc}m_aw_a\geq\frac{\prod(a-b)^2+4\left(\left(\prod r_a\right)^2+r^3\left(\sum r_a\right)^3\right)}{4r^2(4R^2+20Rr-2r^2)}$$

1992. In any Δ ABC, $\frac{R}{2r}\geq\sqrt{1+\frac{3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}}$

Proposed by Bogdan Fuștei - Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $s-a=x, s-b=y$ and $s-c=z\ \therefore\ s=x+y+z\Rightarrow a=y+z, b=z+x$ and $c=x+y$

$$\text{Now, }\frac{s^2}{r^2}=\frac{s^4}{\Delta^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\overset{(a)}{=}\frac{\left(\sum x\right)^3}{xyz}\text{ and }1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod(y+z)}{xyz}\overset{(b)}{=}\frac{xyz+\prod(y+z)}{xyz}$$

$$\sum\frac{b}{a}=\sum\frac{z+x}{y+z}\overset{(c)}{=}\frac{\sum(x+y)^2(y+z)}{\prod(y+z)}\ \therefore\ (a),(b),(c)\Rightarrow\frac{s^2}{r^2}\geq\left(\sum\frac{b}{a}\right)\left(1+\frac{4R}{r}\right)\Leftrightarrow\frac{\left(\sum x\right)^3}{xyz}$$

$$\geq\left[\frac{xyz+\prod(y+z)}{xyz}\right]\left[\frac{\sum(x+y)^2(y+z)}{\prod(y+z)}\right]$$

$$\Leftrightarrow\left\{\prod(y+z)\right\}\left(\sum x\right)^3\geq\left\{xyz+\prod(y+z)\right\}\left(\sum(x+y)^2(y+z)\right)$$

$$\Leftrightarrow\sum x^4y^2+\sum x^3y^3\overset{(i)}{\geq}xyz\left(\sum xy^2\right)+3x^2y^2z^2$$

Now, if $u, v, w > 0$, then

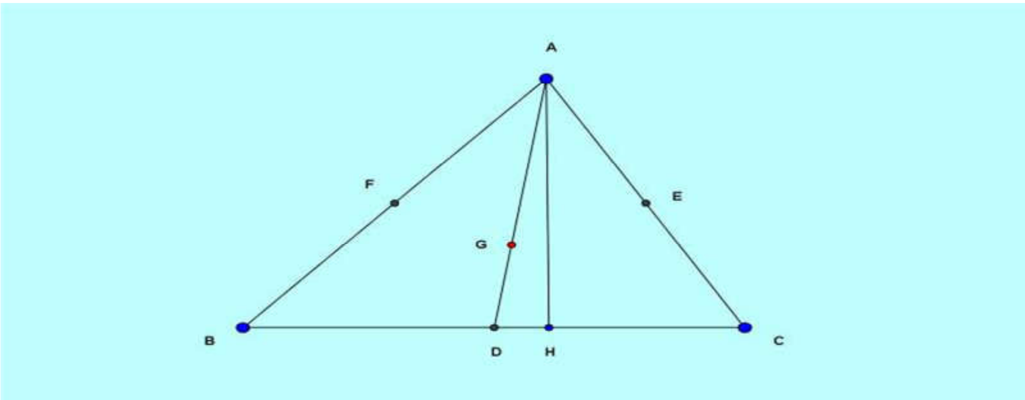
$$: u^3 + u^3 + v^3 \overset{A-G}{\geq} 3u^2v, v^3 + v^3 + w^3 \overset{A-G}{\geq} 3v^2w \text{ and } w^3 + w^3 + u^3 \overset{A-G}{\geq} 3w^2u \text{ and adding these three}:$$

$\sum u^3 \geq \sum u^2v$ and choosing $u = xy, v = yz$ and $w = zx$, we get

$$: \sum x^3y^3 \overset{(ii)}{\geq} xyz\left(\sum xy^2\right) \text{ and } \sum x^4y^2 \underbrace{\overset{A-G}{\geq}}_{(iii)} 3x^2y^2z^2 \therefore (ii) + (iii) \Rightarrow$$

$$(i) \text{ is true} \Rightarrow \frac{s^2}{r^2} \geq \left(\sum \frac{b}{a}\right)\left(1 + \frac{4R}{r}\right) \Rightarrow \frac{s^4}{r^4} \geq \frac{(4R+r)^2}{r^2}\left(\sum \frac{b}{a}\right)^2 \Rightarrow \frac{s^2}{r^2}$$

$$\geq \frac{(4R+r)^2}{s^2}\left(\sum \frac{b}{a}\right)^2 \overset{\text{Trucht}}{\geq} 3\left(\sum \frac{b}{a}\right)^2 \Rightarrow r^2 \overset{(l)}{\leq} \frac{s^2}{3\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)^2}$$



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1993. In any Δ ABC, $\frac{m_a^2}{h_a^2}\geq 1+\frac{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}$

Proposed by Bogdan Fuștei - Romania

Solution by Soumava Chakraborty-Kolkata-India

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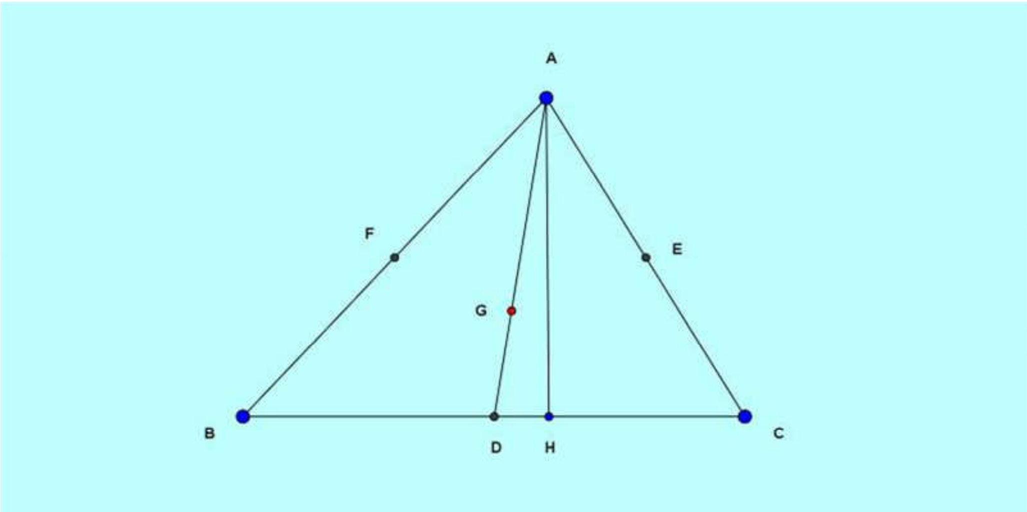

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1994. In any Δ ABC, $\frac{m_a^2}{h_a^2} \geq 1 + \frac{3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}$

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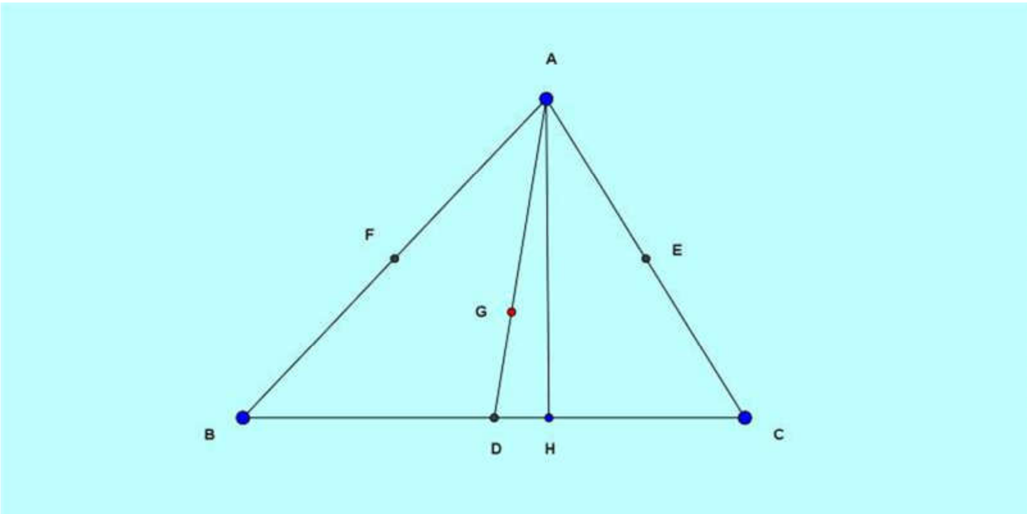

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1995. In $\Delta ABC, n_a$ −Nagel's cevian, the following relationship holds:

$$\sum_{cyc} n_a \geq \frac{2r}{R} \cdot \frac{(5R-r)s^2 - r(4R+r)^2}{s^2 - 6Rr + 3r^2}$$

Proposed by Soumava Chakraborty-Kolkata-India

Solution by Tran Hong-Dong Thap-Vietnam

Lemma 1. In $\Delta ABC, n_a$ −Nagel's cevian, the following relationship holds:

$$\sum_{cyc} n_a^2 = \frac{s^2(3R-r) - r(4R+r)^2}{R}$$

Lemma 2. In $\Delta ABC, n_a$ −Nagel's cevian, the following relationship holds:

$$\sum_{cyc} n_a n_b \geq s^2$$

Let

$$\Omega_1^2 = \left(\sum_{cyc} n_a\right)^2 = \sum_{cyc} n_a^2 + 2\sum_{cyc} n_a n_b \geq$$

$$\geq \frac{s^2(3R-r) - r(4R+r)^2}{R} + 2s^2 \geq \frac{(5R-r)s^2 - r(4R+r)^2}{R}; \ (1)$$

$$\Omega_2^2 = \left(\frac{2r}{R} \cdot \frac{(5R-r)s^2 - r(4R+r)^2}{s^2 - 6Rr + 3r^2}\right)^2; \ (2)$$

From (1),(2) we must to prove:

$$\frac{(5R-r)s^2 - r(4R+r)^2}{R} \geq \frac{4r^2[(5R-r)s^2 - r(4R+r)^2]^2}{R^2(s^2+3r^2-6Rr)^2}$$

$$\Leftrightarrow R(s^2+3r^2-6Rr)^2 \geq 4r^2[(5R-r)s^2 - r(4R+r)^2]$$

$$\Leftrightarrow Rs^4 - (12R^2r + 14Rr^2 - 4r^3)s^2 + 36R^3r^2 + 28R^2r^3 + 41Rr^4 + 4r^5 \geq 0; (3)$$

Let: $\varphi(u) = Ru^2 - (12R^2r + 14Rr^2 - 4r^3)u + 36R^3r^2 + 28R^2r^3 + 41Rr^4 + 4r^5$

$$(u = s^2; 16Rr - 5r^2 \leq u \leq 4R^2 + 4Rr + 3r^2)$$

$$\varphi(u) = 2Ru - (12R^2r + 14Rr^2 - 4r^3) \overset{u\geq 16Rr-5r^2}{\geq} 2R(16Rr - 5r^2)$$

$$-(12R^2r + 14Rr^2 - 4r^3) = 4r(5R^2 - 6Rr + r^2) =$$

$$= 4r(R-r)(5R-r) \overset{R\geq 2r}{\geq} 36r^3 > 0$$

$$\Rightarrow \varphi(u) \uparrow [16Rr - 5r^2, 4R^2 + 4Rr + 3r^2] \Rightarrow \varphi(u) \geq \varphi(16Rr - 5r^2) \overset{(4)}{\geq} 0$$

$$(4) \Leftrightarrow R(16Rr - 5r^2)^2 - (12R^2r + 14Rr^2 - 4r^3)(16Rr - 5r^2) +$$

$$+36R^3r^3 + 28R^2r^3 + 41Rr^4 + 4r^5 \geq 0; \left(t = \frac{R}{r} \geq 2\right)$$

$$\Leftrightarrow t(16t-5)^2 - (12t^2 + 14 - 4)(16t - 5) + 36t^3 + 28t^2 + 41t + 4 \geq 0$$

$$\Leftrightarrow 100t^3 - 296t^2 + 200t - 16 \geq 0$$

$\Leftrightarrow 4(t-2)(25t^2 - 24t + 2) \geq 0$ **which is true, because:**

$$t \geq 2 \Rightarrow t - 2 \geq 0; \ 25t^2 - 24t + 2 = t(25t - 24) + 2 \geq 54 > 0$$

$\Rightarrow (4) \Rightarrow (3)$ is true. Proved.

1996. In $any\ \Delta\ ABC$,

$$1+\left(\frac{1}{a^2}-\frac{1}{(b+c)^2}\right)(b-c)^2 \le \frac{w_a^2}{h_a^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$1+\left(\frac{1}{c^2}-\frac{1}{(a+b)^2}\right)(a-b)^2 \le \frac{w_c^2}{h_c^2} = \frac{4a^2b^2c^2\cos^2\frac{C}{2}}{(a+b)^2 4r^2s^2} = \frac{16R^2r^2s^2\cos^2\frac{C}{2}}{16R^2\cos^2\frac{C}{2}\cos^2\frac{A-B}{2}.r^2s^2}$$

$$= \frac{1}{\cos^2\frac{A-B}{2}}$$

$$\Leftrightarrow \left(\frac{1}{c^2}-\frac{1}{(a+b)^2}\right)(a-b)^2 \overset{(a)}{\le} \frac{1}{\cos^2\frac{A-B}{2}}-1$$

$$\frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} = \frac{\left(16R^2\sin^2\frac{A-B}{2}\sin^2\frac{C}{2}\right)\left(16R^2\cos^2\frac{C}{2}\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right)}{\left(16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\right)\left(16R^2\cos^2\frac{A-B}{2}\cos^2\frac{C}{2}\right)}$$

$$= \frac{1-\cos^2\frac{A-B}{2}}{\cos^2\frac{A-B}{2}} \Rightarrow \frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} \overset{(i)}{=} \frac{1}{\cos^2\frac{A-B}{2}}-1$$

$$\therefore (i) \Rightarrow (a) \Leftrightarrow \frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} \ge \frac{4s(s-c)(a-b)^2}{c^2(a+b)^2} \Leftrightarrow \frac{(s-c)}{r^2} \ge \frac{4s}{c^2} \Leftrightarrow (s-c)c^2 \ge 4sr^2$$

$$= 4(s-a)(s-b)(s-c)$$

$$\Leftrightarrow c^2 \ge (b+c-a)(c+a-b) = c^2-(a-b)^2 \Leftrightarrow (a-b)^2 \ge 0 \rightarrow \text{true}$$

$$\Rightarrow (a) \text{ is true} 1+\left(\frac{1}{c^2}-\frac{1}{(a+b)^2}\right)(a-b)^2 \le \frac{w_c^2}{h_c^2}$$

and analogously $1+\left(\frac{1}{a^2}-\frac{1}{(b+c)^2}\right)(b-c)^2 \le \frac{w_a^2}{h_a^2}$ *(Proved)*

1997. In $\Delta ABC, m(\sphericalangle A) < 90^\circ$ then:

$$s_a \leq w_a.$$

Proposed by Adil Abdullayev, Rahim Shahbazov-Baku-Azerbaijan

Solution by Abdul Hannan-Tezpur-India

$$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} = \frac{\sqrt{bc}}{b+c}\sqrt{(b+c)^2 - a^2}$$

$$s_a = \frac{2bcm_a}{b^2+c^2} = \frac{bc}{b^2+c^2}\sqrt{2b^2+2c^2-a^2}$$

$$w_a^2 - s_a^2 = \frac{bc((b+c)^2-a^2)}{(b+c)^2} - \frac{b^2c^2(2b^2+2c^2-a^2)}{(b^2+c^2)^2}$$

$$\Rightarrow \frac{(w_a^2-s_a^2)(b+c)^2(b^2+c^2)^2}{bc} = ((b+c)^2-a^2) - bc(2b^2+2c^2-a^2)(b+c)^2 =$$

$$= (b+c)^2(b^2+c^2)^2 - a^2(b^2+c^2)^2 - bc(2b^2+2c^2)(b+c)^2 + a^2bc(b+c)^2 =$$

$$= (b+c)^2(b^2+c^2)[(b^2+c^2)-2bc] - a^2[(b^2+c^2)^2 - bc(b+c)^2] =$$

$$= (b+c)^2(b^2+c^2)(b-c)^2 - a^2[(b^4+2b^2c^2+c^4) - (b^3c+2b^2c^2+bc^3)] =$$

$$= (b+c)^2(b^2+c^2)(b-c)^2 - a^2(b-c)(b^3-c^3) =$$

$$= (b-c)^2[(b+c)^2(b^2+c^2) - a^2(b^2+bc+c^2)]$$

Dividing both sides by $2bc(b^2+c^2)$, we obtain:

$$\frac{(w_a^2-s_a^2)(b+c)^2(b^2+c^2)}{b^2c^2} = (b-c)^2\left[\frac{(b+c)^2}{2bc} - \frac{a^2}{2}\cdot\frac{b^2+bc+c^2}{bc(b^2+c^2)}\right] =$$

$$= (b-c)^2\left[1 + \frac{b^2+c^2}{2bc} - \frac{a^2}{2}\cdot\left(\frac{1}{bc} + \frac{1}{b^2+c^2}\right)\right] =$$

$$= (b-c)^2\left[1 + \frac{b^2+c^2-a^2}{2bc} - \frac{a^2}{2(b^2+c^2)}\right] =$$

$$= (b-c)^2\left[1 + cosA - \frac{a^2}{2(b^2+c^2)}\right]$$

This shows that: $w_a > s_a \Leftrightarrow 1 + cosA > \frac{a^2}{2(b^2+c^2)}$

Since $m(\sphericalangle A) < 90^\circ, cosA > 0$

$$\Rightarrow 1 + cosA - \frac{a^2}{2(b^2+c^2)} > 1 + 0 - \frac{a^2}{2(b^2+c^2)} = \frac{4m_a^2}{2(b^2+c^2)} > 0$$

1998. **In ΔABC the following relationship holds:**

$$(m_a + m_b + m_c)\left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right) \le 3\left(\frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c}\right)$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Bogdan Fuştei-Romania

From Chebyshev' inequality, we have: $\sum_{cyc}\frac{m_a}{n_a} \ge \frac{m_a+m_b+m_c}{3}$; (1)

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$$

$$m_a \ge h_a \Rightarrow \sum_{cyc}\frac{1}{n_a} = \frac{1}{r} \ge \sum_{cyc}\frac{1}{m_a}$$

But $n_a \ge m_a$ (and analogs) $\Rightarrow \sum_{cyc}\frac{n_a}{r_a} \ge \sum_{cyc}\frac{m_a}{r_a}$; (2)

$$\frac{m_a + m_b + m_c}{3r} \ge \frac{m_a + m_b + m_c}{3}\left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right); \quad (3)$$

From $(1), (2), (3)$ it follows the desired inequality.

1999. **In ΔABC the following relationship holds:**

$$\left(\frac{R^2}{r^2} + \frac{5R}{2r}\right)\frac{b+c}{a} \cdot sin\frac{A}{2} \ge (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Proposed by Alex Szoros-Romania

Solution by Rahim Shahbazov-Baku-Azerbaijan

$$\frac{b+c}{a} \cdot sin\frac{A}{2} = \frac{sinB + sinC}{sinA} \cdot sin\frac{A}{2} = \frac{2sin\frac{B+C}{2}cos\frac{B-C}{2}}{2sin\frac{A}{2}cos\frac{A}{2}} \cdot sin\frac{A}{2} =$$

$$= cos\frac{B-C}{2} \geq cos^2\frac{B-C}{2} \geq \frac{2r}{R}$$

We must show that: $\left(\frac{R^2}{r^2}+\frac{5R}{2r}\right)\frac{2r}{R} \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$$\Leftrightarrow 2s\cdot\frac{s^2+r^2+4Rr}{4Rrs} \leq \frac{2R}{r}+5$$

$$\Leftrightarrow s^2+r^2+4Rr \leq 4R^2+10Rr$$

$$\Leftrightarrow s^2 \leq 4R^2+6Rr-r^2; \ \ (1)$$

$$s^2 \leq 4R^2+4Rr+3r^2 \ (Gerretsen)$$

Hence, we must prove: $s^2 \leq 4R^2+4Rr+3r^2 \leq 4R^2+6Rr-r^2$

$\Leftrightarrow R > 2r \ (Euler) \Rightarrow (1)$ is true.

2000. In ΔABC the following relationship holds:

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} \geq \frac{3(a^2+b^2+c^2)}{a+b+c}$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Marian Dincă-Romania

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} = \frac{(ab)^2+(bc)^2+(ca)^2}{abc} = \frac{(ab+bc+ca)^2-2abc(a+b+c)}{abc} =$$

$$= \frac{(s^2+r^2+4Rr)^2-4s(4Rrs)}{4Rrs}$$

$$\frac{3(a^2+b^2+c^2)}{a+b+c} = \frac{3[(a+b+c)^2-2(ab+bc+ca)]}{a+b+c} = \frac{3(2s^2-2r^2-8Rr)}{2s}$$

$$\frac{(s^2+r^2+4Rr)^2-4s(4Rrs)}{4Rrs} \geq \frac{3(2s^2-2r^2-8Rr)}{2s}$$

$$(s^2+r^2+4Rr)^2-4s(4Rrs) \geq 6Rrs(2s^2-2r^2-8Rr)$$

$$(s^2+r^2+4Rr)^2-28s^2Rr+12Rr(r^2+4Rr) \geq 0$$

$$s^4+2s^2(r^2+4Rr)+(r^2+4Rr)^2-28s^2Rr+12Rr(r^2+4Rr) \geq 0$$

$$s^4+s^2(2r^2-20Rr)+(r^2+4Rr)^2+12Rr(r^2+4Rr) \geq 0$$

$$(s^2-10Rr+r^2)^2-(r^2-10Rr)^2+(r^2+4Rr)^2+12Rr(r^2+4Rr)\geq 0$$

Using Blundon inequality:

$$s^2-10Rr+r^2\geq 2R^2-2(R-2r)\sqrt{R^2-2Rr}>0$$

It will suffice to prove for the isosceles triangle, because $s\geq s_1$,where s_1 is the semiperimeter of an isosceles triangle having the same R and r returning to algebraic inequality. Let $a+b+c=1$, because is homogeneous.

$Let\ a=b=t, c=1-2t$, because $a=b<c$ and $a+b>c$ result

$t\in\left[\frac{1}{4},\frac{1}{3}\right]$, the inequality $\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq\frac{3(a^2+b^2+c^2)}{a+b+c}$ becomes:

$$\frac{t^2}{1-2t}+2(1-2t)\geq 3(2t^2+(1-2t)^2)$$

$$\Leftrightarrow t^2+2(1-2t)^2-3(12t)(t^2+(1-2t)^2)\geq 0$$

$$\Leftrightarrow (3t-1)^2(4t-1)\geq 0$$

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To be continued!

Daniel Sitaru