

*RMM - Cyclic Inequalities Marathon 601 - 700*

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**601. If  $x, y, z > 0$  then:**

$$\frac{2xy}{(x+y)^2} + \frac{2xz}{(x+z)^2} + \frac{2yz}{(y+z)^2} \leq \frac{1}{2} + \frac{8xyz}{(x+y)(y+z)(z+x)}$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned}
 & \frac{2xy}{(x+y)^2} + \frac{2xz}{(x+z)^2} + \frac{2yz}{(y+z)^2} \leq \frac{1}{2} + \frac{8xyz}{(x+y)(y+z)(z+x)} \\
 \Leftrightarrow & \frac{-4xy}{(x+y)^2} + \frac{-4xz}{(x+z)^2} + \frac{-4yz}{(y+z)^2} \geq \frac{-16xyz}{(x+y)(y+z)(z+x)} - 1 \\
 \Leftrightarrow & 1 - \frac{4xy}{(x+y)^2} + 1 - \frac{4xz}{(x+z)^2} + 1 - \frac{4yz}{(y+z)^2} + \frac{16xyz - 2(x+y)(y+z)(z+x)}{(x+y)(y+z)(z+x)} \geq 0 \\
 \Leftrightarrow & \frac{(x-y)^2}{(x+y)^2} + \frac{(x-z)^2}{(x+z)^2} + \frac{(y-z)^2}{(y+z)^2} - \frac{2x(y-z)^2 + 2y(x-z)^2 + 2z(x-y)^2}{(x+y)(y+z)(z+x)} \geq 0 \\
 \Leftrightarrow & \sum_{cyc} \frac{(y-z)^2}{(y+z)^2} \left( \frac{1}{y+z} - \frac{2x}{(x+y)(x+z)} \right) \geq 0 \\
 \Leftrightarrow & \sum_{cyc} \left( \frac{(y-z)^2}{(y+z)^2} \cdot \frac{(x-y)(x-z)}{(x+y)(y+z)(z+x)} \right) \geq 0 \\
 \Leftrightarrow & -\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)} \cdot \frac{(x-y)(y-z)(x-z)}{(x+y)(y+z)(z+x)} \geq 0 \\
 \Leftrightarrow & \left( \frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)} \right)^2 \geq 0
 \end{aligned}$$

*Which is true. Equality  $\Leftrightarrow x = y = z$ . Let:  $x = a; y = b; z = c \Rightarrow (*)$  is true. Proved.*

**602. If  $a, b, c > 0, abc = 1$  then:**

$$\sqrt[4]{\frac{b^2 + c^2}{2a}} + \sqrt[4]{\frac{c^2 + a^2}{2b}} + \sqrt[4]{\frac{a^2 + b^2}{2c}} \leq a + b + c$$

*Proposed by George Apostolopoulos-Messolonghi-Greece*



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**Solution 1 by Rahim Shahbazov-Baku-Azerbaijan**

$$\sqrt[4]{\frac{b^2 + c^2}{2a}} + \sqrt[4]{\frac{c^2 + a^2}{2b}} + \sqrt[4]{\frac{a^2 + b^2}{2c}} \leq a + b + c \quad (1)$$

$$\begin{aligned} \frac{b^2 + c^2}{2a} &= \frac{bc(b^2 + c^2)}{2} = \frac{8bc(b^2 + c^2)}{16} = \frac{4(2bc)(b^2 + c^2)}{16} \leq \frac{(b+c)^4}{16} \stackrel{(1)}{\Rightarrow} \\ LHS &\leq \frac{b+c}{2} + \frac{c+a}{2} + \frac{a+b}{2} = a + b + c \end{aligned}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $a, b, c > 0, abc = 1$  we give  $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$

Consider:

$$\begin{aligned} \sqrt[4]{\frac{b^2 + c^2}{2a}} + \sqrt[4]{\frac{c^2 + a^2}{2b}} + \sqrt[4]{\frac{a^2 + b^2}{2c}} &= \sqrt[4]{\frac{b^3c + bc^3}{2}} + \sqrt[4]{\frac{c^3a + a^3c}{2}} + \sqrt[4]{\frac{a^3b + ab^3}{2}} \leq 0 \\ \sqrt[4]{\frac{b^2 + c^2}{2}} &\leq \frac{b+c}{2} \end{aligned}$$

$$Iff \quad \frac{b^3c + bc^3}{2} \leq \frac{b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4}{16}$$

$$4(b^3c + bc^3) \leq b^4 + c^4 + 6b^2c^2 \Leftrightarrow 4\left(\frac{b}{c} + \frac{c}{b}\right) \leq \frac{b^2}{c^2} + \frac{c^2}{b^2} + 6$$

$$\Leftrightarrow 4\left(\frac{y}{z} + \frac{z}{y}\right) \leq \left(\frac{y}{z}\right)^2 + \left(\frac{z}{y}\right)^2 + 6 \Leftrightarrow 4\left(\frac{xy}{z^2} + \frac{z^2}{xy}\right) \leq \frac{(xy)^2}{z^4} + \frac{z^4}{(xy)^2} + 6$$

$$\Leftrightarrow 4((xy)^3z^2 + z^6xy) \leq (xy)^4 + z^8 + 6z^4(xy)^2$$

$$\Leftrightarrow 3(z^2 - xy)(z^4xy - (xy)^2z^2) \leq (z^2 - xy)(z^6 - (xy)^3)$$

$$\Leftrightarrow 3(z^2 - xy)(z^2xy) \leq (z^2 - xy)^2(z^4 + z^2xy + (xy)^2) \text{ true.}$$

Then:  $\sqrt[4]{\frac{b^2+c^2}{2}} \leq \frac{b+c}{2}$  and analogs.

$$So, \sqrt[4]{\frac{b^2+c^2}{2a}} + \sqrt[4]{\frac{c^2+a^2}{2b}} + \sqrt[4]{\frac{a^2+b^2}{2c}} \leq a + b + c$$



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603. If  $a, b, c > 0$  then:

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 2 \cdot \sqrt{(ab+bc+ca)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

When does equality holds?

*Proposed by Nguyen Van Canh-Vietnam*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\begin{aligned} \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} &\geq 2 \cdot \sqrt{(ab+bc+ca)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)} \Leftrightarrow \\ \left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right)^2 &\geq 4 \left( \sum_{cyc} ab \cdot \sum_{cyc} \frac{1}{a^2} \right) \Leftrightarrow \\ (ab(a+b) + bc(b+c) + ca(c+a)) &\geq 4 \sum_{cyc} a \cdot \sum_{cyc} a^2 b^2 \Leftrightarrow \\ \sum_{cyc} a^2 b^2 (a+b)^2 + 2abc \left( \sum_{cyc} a(a+b)(a+c) \right) &\geq 4 \left( \sum_{cyc} a^3 b^3 + abc \sum_{cyc} ab(a+b) \right) \end{aligned}$$

Which is true because:

$$\begin{aligned} \sum_{cyc} a^2 b^2 (a+b)^2 - 4 \sum_{cyc} a^3 b^3 &= \sum_{cyc} a^2 b^2 (a-b)^2 \geq 0 \\ 2abc \left( \sum_{cyc} a(a+b)(a+c) \right) - 4abc \sum_{cyc} ab(a+b) &= \\ = 2abc (\sum_{cyc} a^3 + 3abc - \sum_{cyc} ab(a+b)) &\geq 0 \text{ true by Schur's inequality.} \end{aligned}$$

Proved. Equality for  $a = b = c$ .

604. If  $a_1, a_2, \dots, a_n > 0$  then:

$$\prod_{i=1}^n \left(1 + a_i^{1+a_i^{1+a_i}}\right) \geq 2^n \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n} \sum_{i=1}^n a_i}$$

*Proposed by Florică Anastase-Romania*



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**Solution 1 by Tran Hong-Dong Thap-Vietnam**

We have inequality:  $1 + \alpha^{1+\alpha^{1+\alpha}} \geq 2\alpha^\alpha, \forall \alpha > 0$

$$\begin{aligned} \prod_{i=1}^n \left(1 + a_i^{1+a_i^{1+a_i}}\right) &\geq \prod_{i=1}^n (2a_i^{a_i}) \geq 2^n \prod_{i=1}^n a_i^{a_i} \stackrel{AM-GM}{\geq} 2^n \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^{a_1+a_2+\dots+a_n} \\ &\stackrel{AM-GM}{\geq} 2^n (\sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n})^{a_1+a_2+\dots+a_n} = 2^n (a_1 \cdot a_2 \cdot \dots \cdot a_n)^{\frac{a_1+a_2+\dots+a_n}{n}} = RHS \end{aligned}$$

*Proved.*

**Solution 2 by proposer**

$$\begin{aligned} 1 + a_i^{1+a_i} &= 1 + (1 + a_i - 1)^{1+a_i} \stackrel{Benoulli}{\geq} 1 + a_i^2 \rightarrow \\ 1 + a_i^{1+a_i^{1+a_i}} &\geq 1 + a_i^{1+a_i^2} \stackrel{AM-GM}{\geq} 2a_i^{a_i} \rightarrow \prod_{i=1}^n \left(1 + a_i^{1+a_i^{1+a_i}}\right) \geq 2^n \prod_{i=1}^n a_i^{a_i} \dots \dots (1^\circ) \end{aligned}$$

*We must show:*

$$\prod_{i=1}^n a_i^{a_i} \geq \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n} \sum_{i=1}^n a_i} \leftrightarrow \sum_{i=1}^n a_i \log(a_i) \geq \frac{1}{n} \left(\sum_{i=1}^n a_i\right) \left(\sum_{i=1}^n \log(a_i)\right) \text{ true,}$$

*Cebyshev inequalities for sequences  $(a_i)_{i \geq 1}, (\log(a_i))_{i \geq 1}, \dots, (2^\circ)$*

*From  $(1^\circ), (2^\circ)$  we have:*

$$\prod_{i=1}^n \left(1 + a_i^{1+a_i^{1+a_i}}\right) \geq 2^n \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n} \sum_{i=1}^n a_i}$$

**605. If  $a, b, c > 0$  then:**

$$\frac{a^3 + b^3 + c^3}{abc} + 36 \left( \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \right) \geq 57$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned} \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} &= \frac{a^2}{a^2+ab} + \frac{b^2}{b^2+bc} + \frac{c^2}{c^2+ca} \stackrel{CBS}{\geq} \frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca} \\ \frac{a^3+b^3+c^3}{abc} + 36 \left( \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \right) &\geq \frac{a^3+b^3+c^3}{abc} + 36 \cdot \frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca} \stackrel{(1)}{\geq} 57 \end{aligned}$$



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*Let:  $p = a + b + c; q = ab + bc + ca; r = abc$*

$$(1) \Leftrightarrow \frac{p^3 - 3pq + 3r}{r} + 36 \cdot \frac{p^2}{p^2 - 2q + q} \geq 57$$

$$\frac{p(p^2 - 3q)}{r} + 36 \cdot \frac{p^2}{p^2 - q} \geq 54$$

*But:  $(a + b + c)(ab + bc + ca) \geq 9abc \Rightarrow pq \geq 9r \Rightarrow r \leq \frac{pq}{9}$*

$$\frac{9p(p^2 - 3q)}{pq} + 36 \cdot \frac{p^2}{p^2 - q} \stackrel{(2)}{\geq} 54$$

$$(2) \Leftrightarrow \frac{9(p^2 - 3q)}{q} + 36 \left(1 + \frac{q}{p^2 - q}\right) \geq 54$$

$$\frac{9(p^2 - 3q)}{q} + \frac{36q}{p^2 - q} \geq 18 \Leftrightarrow \frac{p^2 - 3q}{q} + \frac{4q}{p^2 - q} \geq 2$$

$$\frac{p^2}{q} + \frac{4q}{p^2 - q} \geq 5 \Leftrightarrow p^2(p^2 - q) + 4q^2 \geq 5q(p^2 - q)$$

$\Leftrightarrow (p^2 - 3q)^2 \geq 0 \Rightarrow (2) \text{ is true} \Rightarrow (1) \text{ is true. Proved. Equality} \Leftrightarrow a = b = c.$

**606. If  $x, y, z > 0$  then:**

$$\sqrt{\frac{x(x+z)}{y(y+z)}} + \sqrt{\frac{y(y+x)}{z(z+x)}} + \sqrt{\frac{z(z+y)}{x(x+y)}} + \frac{8xyz}{(x+y)(y+z)(z+x)} \geq 4$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution by Tran Hong-Dong Thap-Vietnam**

*Let:  $a = \sqrt{\frac{x}{y+z}}, b = \sqrt{\frac{y}{z+x}}, c = \sqrt{\frac{z}{x+y}}$*

$$a^2 = \frac{x}{y+z}, b^2 = \frac{y}{z+x}, c^2 = \frac{z}{x+y}$$

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} = 2 \Rightarrow 2a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2 = 1$$

$\Rightarrow \exists \Delta XYZ \text{ acute such that: } ab = \cos Z; bc = \cos X; ca = \cos Y. \text{ Hence}$



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$$\sqrt{\frac{x(x+z)}{y(y+z)}} + \sqrt{\frac{y(y+x)}{z(z+x)}} + \sqrt{\frac{z(z+y)}{x(x+y)}} + \frac{8xyz}{(x+y)(y+z)(z+x)} \geq 4$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 8(abc)^2 \geq 4 \Leftrightarrow$$

$$\frac{\cos Y}{\cos X} + \frac{\cos Z}{\cos Y} + \frac{\cos X}{\cos Z} + 8\cos X \cos Y \cos Z \stackrel{(1)}{\geq} 4$$

$$\frac{\cos Y}{\cos X} + \frac{\cos Z}{\cos Y} + \frac{\cos X}{\cos Z} \geq \frac{9(\cos^2 X + \cos^2 Y + \cos^2 Z)}{(\cos X + \cos Y + \cos Z)^2}$$

$$\begin{aligned} & 0 < \cos X + \cos Y + \cos Z \leq \frac{3}{2} \\ & \stackrel{(1)}{\geq} \frac{9(\cos^2 X + \cos^2 Y + \cos^2 Z)}{\left(\frac{3}{2}\right)^2} \end{aligned}$$

$$= 4(\cos^2 X + \cos^2 Y + \cos^2 Z) = 4(1 - 2\cos X \cos Y \cos Z) \Rightarrow (*) \text{ is true. Proved.}$$

**607. If  $a, b, c > 0$  prove:**

$$\frac{(a^2 + a + 1)^{\sqrt{3}}(b^2 + b + 1)^{\sqrt{3}}(c^2 + c + 1)^{\sqrt{3}}}{e^{2a} \cdot e^{2b} \cdot e^{2c}} \leq 1$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Adrian Popa-Romania**

$$a, b, c > 0$$

$$\frac{(a^2 + a + 1)^{\sqrt{3}}(b^2 + b + 1)^{\sqrt{3}}(c^2 + c + 1)^{\sqrt{3}}}{e^{2a} \cdot e^{2b} \cdot e^{2c}} \leq 1 \Leftrightarrow$$

$$\prod_{cyc} (a^2 + a + 1)^{\sqrt{3}} \leq \prod_{cyc} e^{2a} \quad (*)$$

$$\sqrt{3} \sum_{cyc} \log(a^2 + a + 1) \leq 2 \sum_{cyc} a$$

*Let:  $f: (0, \infty) \rightarrow \mathbb{R}; f(x) = \sqrt{3} \log(x^2 + x + 1) - 2x$*

$$f'(x) = \frac{-2x^2 + (2\sqrt{3} - 2)x + \sqrt{3} - 2}{x^2 + x + 1}$$



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$f'(x) < 0, \forall x \geq 0$  then

$$\begin{cases} f \downarrow \\ f(0) = 0 \end{cases} \Rightarrow f(x) \leq 0, \forall x \geq 0 \Rightarrow \sqrt{3} \log(x^2 + x + 1) \leq 2x$$

$$\begin{cases} \sqrt{3} \log(a^2 + a + 1) \leq 2a \\ \sqrt{3} \log(b^2 + b + 1) \leq 2b \\ \sqrt{3} \log(c^2 + c + 1) \leq 2c \end{cases} \Rightarrow (*)$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

First, we prove:  $\therefore 2e^{2x} \geq 2 + 4x + 4x^2 \quad (1)$

$$\therefore \sqrt{3}(2x+1)(x^2+x+1)^{\sqrt{3}-1} \leq (3+\sqrt{3})x + \sqrt{3} \quad (2)$$

**Proof:**

Let:  $\varphi(x) = (3+\sqrt{3})x + \sqrt{3} - \sqrt{3}(2x+1)(x^2+x+1)^{\sqrt{3}-1}, x \geq 0$

$$\varphi'(x) = 2\sqrt{3}(x^2+x+1)^{\sqrt{3}-1} + \sqrt{3}(\sqrt{3}-1)(2x+1)^2(x^2+x+1)^{\sqrt{3}-2} - \sqrt{3} - 3 \stackrel{x \geq 0}{\geq} 0$$

Then  $\varphi(x) \nearrow$  on  $[0, \infty)$   $\Rightarrow \varphi(x) \geq \varphi(0) = 0 \Rightarrow (2)$  true

Let:  $g(x) = 2(2e^{2x} - 2x^2 - 2x - 1), x \geq 0$

$$g'(x) = 2(2e^{2x} - 4x - 2) = 4(e^{2x} - 2x - 1) \geq 0$$

Then  $g(x) \nearrow$  on  $[0, \infty)$   $\Rightarrow g(x) \geq g(0) = 0 \Rightarrow (1)$  true

Now, we must show that:  $e^{2x} \geq (x^2 + x + 1)^{\sqrt{3}}, \forall x \geq 0 \quad (3)$

Let:  $\varphi(x) = e^{2x} - (x^2 + x + 1)^{\sqrt{3}}, \forall x \geq 0$

$$\varphi'(x) = 2e^{2x} - \sqrt{3}(2x+1)(x^2+x+1)^{\sqrt{3}-1}$$

$$\geq 4x^2(1-\sqrt{3})x + 2 - \sqrt{3} > 0, \forall x \geq 0$$

$\varphi(x) \geq \varphi(0) = 0 \Rightarrow (3)$  true. Hence:  $(a^2 + a + 1)^{\sqrt{3}} \leq e^{2a}$

$$\prod_{cyc} (a^2 + a + 1)^{\sqrt{3}} \leq \prod_{cyc} e^{2a}$$

Equality for  $a = b = c = 0$

**608. In  $\Delta ABC$  the following relationship holds:**

$$\frac{(3a+b+c)^2}{2a+b+2c} + \frac{(a+3b+c)^2}{2a+2b+c} + \frac{(a+b+3c)^2}{a+2b+2c} \leq \frac{19 \cdot (a^2 + b^2 + c^2) + 26 \cdot (ab + bc + ca)}{3 \cdot (a + b + c)}$$

*Proposed by George Florin Ţerban-Romania*



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**Solution by Rahim Shahbazov-Baku-Azerbaijan**

$$\begin{cases} 2a + b + c = x \\ a + 2b + c = y \\ a + b + 2c = z \end{cases} \Rightarrow \begin{cases} b = \frac{2y + 2z - 3x}{5} \\ c = \frac{2x + 2z - 3y}{5} \\ a = \frac{2x + 2y - 3z}{5} \end{cases}$$

*Inequality becomes:*

$$\begin{aligned} \frac{(x+y-z)^2}{x} + \frac{(y+z-x)^2}{y} + \frac{(x+z-y)^2}{z} &\leq \frac{23(x^2 + y^2 + z^2) - 14(xy + yz + zx)}{3(x+y+z)} \\ x + 2(y-z) + \frac{(y-z)^2}{x} + y + 2(z-x) + \frac{(z-x)^2}{y} + z + 2(x-y) + \frac{(x-y)^2}{z} \\ &\leq \frac{23(x^2 + y^2 + z^2) - 14(xy + yz + zx)}{3(x+y+z)} \\ \frac{(y-z)^2}{x} + \frac{(z-x)^2}{y} + \frac{(x-y)^2}{z} &\leq \frac{20(x^2 + y^2 + z^2 - xy - yz - zx)}{3(x+y+z)} \\ \sum_{cyc} \frac{(y-z)^2}{x} &\leq \sum_{cyc} \frac{10(y-z)^2}{3(x+y+z)} \\ \sum_{cyc} (y-z)^2 \left[ \frac{10}{3(x+y+z)} - \frac{1}{x} \right] &\geq 0 \\ \sum_{cyc} (y-z)^2 \frac{7x - 3y - 3z}{3x(x+y+z)} &\geq 0 \\ 7x - 3y - 3z &= 5(a + c - b) > 0 \end{aligned}$$

**609. If  $a, b, c > 0$  then:**

$$\frac{(3a+2b+c+6)(3b+2c+a+6)(3c+2a+b+6)}{(a+1)(b+1)(c+1)} \geq 216$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by George Florin Șerban-Romania**

*Denote:  $a+1 = x; b+1 = y; c+1 = z; x, y, z > 0 \Rightarrow$*

$$\frac{(3x+2y+z)(x+3y+2z)(2x+y+3z)}{xyz} \geq 216$$



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$$3x + 2y + z = x + x + x + y + y + z \stackrel{Am-Gm}{\geq} 6 \cdot \sqrt[6]{x^3 y^2 z}$$

$$x + 3y + 2z = x + y + y + y + z + z \stackrel{Am-Gm}{\geq} 6 \cdot \sqrt[6]{x y^3 z^2}$$

$$2x + y + 3z = x + x + y + z + z + z \stackrel{Am-Gm}{\geq} 6 \cdot \sqrt[6]{x^2 y z^3}$$

$$\prod_{cyc} (3x + 2y + z) \geq 6^3 \cdot \sqrt[6]{x^6 y^6 z^6} = 216xyz$$

$$\frac{\prod (3x + 2y + z)}{xyz} \geq 216$$

*Equality for  $a = b = c$*

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

*Denote:  $a + 1 = x; b + 1 = y; c + 1 = z; x, y, z > 0 \Rightarrow$*

$$a = x - 1; b = y - 1; c = z - 1$$

$$Inequality \Leftrightarrow \frac{\prod [3(x-1) + 2(y-1) + z-1 + 6]}{xyz} \geq 216$$

$$\Leftrightarrow \prod_{cyc} (3x + 2y + z) \geq 216xyz \dots (*)$$

$$3x + 2y + z = x + x + x + y + y + z \stackrel{Am-Gm}{\geq} 6 \cdot \sqrt[6]{x^3 y^2 z}$$

$$x + 3y + 2z = x + y + y + y + z + z \stackrel{Am-Gm}{\geq} 6 \cdot \sqrt[6]{x y^3 z^2}$$

$$2x + y + 3z = x + x + y + z + z + z \stackrel{Am-Gm}{\geq} 6 \cdot \sqrt[6]{x^2 y z^3}$$

$$\prod_{cyc} (3x + 2y + z) \geq (6 \cdot \sqrt[6]{x^3 y^2 z}) (6 \cdot \sqrt[6]{x y^3 z^2}) (6 \cdot \sqrt[6]{x^2 y z^3})$$

$$= 216 \cdot \sqrt[6]{x^6 y^6 z^6} = 216xyz$$

*Equality for  $a = b = c$*

**Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $a, b, c > 0$ , we get*

$$\frac{(3a + 2b + c + 6)(3b + 2c + a + 6)(3c + 2a + b + 6)}{(a + 1)(b + 1)(c + 1)} \geq$$



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$$\begin{aligned}
 &\geq \frac{\left(\sqrt[3]{(3a+3)(3b+3)(3c+3)} + \sqrt[3]{(2a+2)(2b+2)(2c+2)} + \sqrt[3]{(a+1)(b+1)(c+1)}\right)^3}{(a+1)(b+1)(c+1)} \\
 &= \frac{\left(3\sqrt[3]{(a+1)(b+1)(c+1)} + 2\sqrt[3]{(a+1)(b+1)(c+1)} + \sqrt[3]{(a+1)(b+1)(c+1)}\right)^3}{(a+1)(b+1)(c+1)} \\
 &= \frac{\left(6\sqrt[3]{(a+1)(b+1)(c+1)}\right)^3}{(a+1)(b+1)(c+1)} = \frac{216(a+1)(b+1)(c+1)}{(a+1)(b+1)(c+1)} = 216
 \end{aligned}$$

**610. If  $x, y, z > 0$  then:**

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9 \cdot \sqrt{\frac{x^2 + y^2 + z^2}{xy + yz + zx}}$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned}
 &(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9 \cdot \sqrt{\frac{x^2 + y^2 + z^2}{xy + yz + zx}} \quad (*) \\
 &(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) - 9 \geq 9 \cdot \sqrt{\frac{x^2 + y^2 + z^2}{xy + yz + zx}} - 9 \\
 &\frac{(y-z)^2}{yz} + \frac{(z-x)^2}{zx} + \frac{(x-y)^2}{xy} \geq \frac{9[(x-y)^2 + (y-z)^2 + (z-x)^2]}{2 \cdot \sqrt{xy + yz + zx} (\sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx})} \\
 &\sum_{cyc} \left( \frac{1}{xy} - \frac{9}{2 \cdot \sqrt{xy + yz + zx} (\sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx})} \right) (x-y)^2 \geq 0 \\
 &\sum_{cyc} \frac{(2 \cdot \sqrt{xy + yz + zx} (\sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx}) - 9xy)}{xy \cdot 2 \cdot \sqrt{xy + yz + zx} (\sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx})} \cdot (x-y)^2 \geq 0 \\
 &\sum_{cyc} [2 \cdot \sqrt{xy + yz + zx} (\sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx}) - 9xy] \cdot (x-y)^2 \geq 0
 \end{aligned}$$

Let:  $S = \sum_{cyc} [2 \cdot \sqrt{xy + yz + zx} (\sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx}) - 9xy] \cdot (x-y)^2$



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$$S_z = 2 \cdot \sqrt{xy + yz + zx} \left( \sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx} \right) - 9xy$$

$$S_y = 2 \cdot \sqrt{xy + yz + zx} \left( \sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx} \right) - 9zx$$

$$S_x = 2 \cdot \sqrt{xy + yz + zx} \left( \sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx} \right) - 9yz$$

$$\text{Now, we have: } \sqrt{x^2 + y^2 + z^2} \geq \sqrt{xy + yz + zx}$$

$$S_x + S_y + S_z \geq 12(xy + yz + zx) - 9(xy + yz + zx) = 3(xy + yz + zx) \geq 0; (*)$$

$$S_x S_y + S_y S_z + S_z S_x = 12t^2 - 36(xy + yz + zx) + 81(x^2y^2 + y^2z^2 + z^2x^2) = \varphi(t),$$

$$t = \sqrt{xy + yz + zx} \left( \sqrt{x^2 + y^2 + z^2} + \sqrt{xy + yz + zx} \right) > 0$$

$$\varphi'(t) = 0 \Rightarrow t_0 = \frac{3}{2}(xy + yz + zx) \Rightarrow \varphi(t) \geq \varphi(t_0)$$

$$12 \cdot \frac{3}{4}(xy + yz + zx)^2 - 36 \cdot \frac{3}{2}(xy + yz + zx)^2 + 81(x^2y^2 + y^2z^2 + z^2x^2)$$

$$= 27(xy + yz + zx)^2 - 54(xy + yz + zx)^2 + 81(x^2y^2 + y^2z^2 + z^2x^2)$$

$$\geq 27(xy + yz + zx)^2 - 54(xy + yz + zx)^2 + 27(xy + yz + zx)^2 = 0 \Rightarrow$$

$$S_x S_y + S_y S_z + S_z S_x \geq 0; (**)$$

*From (\*), (\*\*)  $\Rightarrow S \geq 0$ . Proved.*

*Equality  $\Leftrightarrow x = y = z$ .*

**Solution 2 by Michael Sterghiou-Greece**

*Let:  $(p, q, r) = (\sum x, \sum xy, \prod x)$ .*

*Wlog we can assume  $p = 3$  due to (1) being homogeneous.*

*With  $\sum x^2 = p^2 - 2q = 9 - 2q$  (1) reduces to:  $q^3 - 9(9 - 2q)r^2 = f(r) \geq 0$  (2)*

*Now  $f(r)$  is decreasing function of  $r$  therefore it suffices to show (2) for maximum  $r$ .*

*According to V.Cirtoaje theorem, and assuming Wlog  $z \leq y \leq x$ , then,  $r$  becomes*

*maximal where  $y = z$ .*

*In such case  $y \leq 1, x = 3 - 2y, q = y^2 + 2y(3 - 2y), r = y^2(3 - 2y)$  and (2) becomes*

*after some computation  $-27(y - 1)^2 y^3 \underbrace{(8y^3 - 24y^2 + 23y - 8)}_{h(y)} \geq 0$  (3)*

*Now  $h(y)$  can be written  $8(y^3 - 3y^2 + 3y - 1) - y = (y - 1)^3 - y < 0, y \leq 1$*

*And (3) is clearly positive or zero for  $y = 1$ .*



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*Equality for  $x = y = z = 1$ . Done!*

**611. If  $x, y, z > 0, x + y + z = 1, n \geq 2$  then:**

$$\frac{x}{\sqrt{nx+y}} + \frac{y}{\sqrt{ny+z}} + \frac{z}{\sqrt{nz+x}} \leq \sqrt{\frac{3}{n+1}}$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

$$LHS = \sum_{cyc} \frac{x}{\sqrt{nx+y}} = \sum_{cyc} \sqrt{x} \left( \sqrt{\frac{x}{nx+y}} \right)^{BCS} \geq \sqrt{x+y+z} \cdot \sqrt{\frac{x}{nx+y} + \frac{y}{ny+z} + \frac{z}{nz+x}}$$

$$= \sqrt{\frac{x}{nx+y} + \frac{y}{ny+z} + \frac{z}{nz+x}}$$

$$Put \Phi = \frac{x}{nx+y} + \frac{y}{ny+z} + \frac{z}{nz+x} \leq \frac{3}{n+1} \Leftrightarrow$$

$$(n+1)[x(ny+z)(nz+x) + y(nx+y)(nz+x) + z(nx+y)(ny+z)]$$

$$\leq 3(nx+y)(ny+z)(nz+x) \Leftrightarrow$$

$$(n^2 - 2n)(x^2y + y^2z + z^2x) + (2n-1)(xy^2 + yz^2 + zx^2) \geq (3n^2 - 1)xyz; (*)$$

*Because:  $n \geq 2 \Rightarrow n^2 - 2n \geq 0, 2n - 1 \geq 3$*

$$(n^2 - 2n)(x^2y + y^2z + z^2x) \stackrel{Am-Gm}{\leq} 3(n^2 - 2n) \sqrt[3]{(xyz)^3} = 3(n^2 - 2n)xyz; (1)$$

$$(2n-1)(xy^2 + yz^2 + zx^2) \stackrel{Am-Gm}{\leq} 3(2n-1) \sqrt[3]{(xyz)^3} = (2n-1)xyz; (2)$$

$$\stackrel{(1)+(2)}{\implies} (n^2 - 2n)(x^2y + y^2z + z^2x) + (2n-1)(xy^2 + yz^2 + zx^2) \geq 3(n^2 - 2n)xyz$$

*$\Rightarrow (*) \text{ true.}$*

$$\Rightarrow \Phi \leq \frac{3}{n+1} \Rightarrow LHS \leq \sqrt{\Phi} \leq \sqrt{\frac{3}{n+1}}. \text{Proved.}$$

*Equality  $\Leftrightarrow x = y = z = 1$*

**Solution 2 by Michael-Stergiou-Greece**

$$\sum_{cyc} \frac{x}{\sqrt{nx+y}} \leq \sqrt{\frac{3}{n+1}} \quad (1)$$



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$$(1) \text{ can be rewritten as } \sum_{\text{cyc}} \sqrt{\frac{x^2}{nx+y}} \leq \sqrt{\frac{3}{n+1}} \quad (2)$$

*By concavity of  $f(t) = \sqrt{t}$  on  $(0, \infty)$*

$$\text{LHS} \stackrel{\text{Jensen}}{\leq} 3 \cdot \sqrt{\frac{\left(\frac{\sum x}{3}\right)^2}{n\frac{\sum x}{3} + \frac{\sum y}{3}}} = 3 \cdot \sqrt{\frac{\frac{1}{9}}{\frac{n+1}{3}}} = \sqrt{\frac{3}{n+1}}$$

**612. If  $a, b, c > 0$  then:**

$$\left( \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \right) \sqrt{\frac{a^2 + b^2 + c^2}{ab + bc + ca}} \geq \frac{3}{2}$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution by Soumitra Mandal-Chandahar Nagore-India**

$$\begin{aligned} & \sqrt{\frac{a^2 + b^2 + c^2}{ab + bc + ca}} \sum_{\text{cyc}} \frac{a}{a+b} = \sqrt{\frac{a^2 + b^2 + c^2}{ab + bc + ca}} \sum_{\text{cyc}} \frac{a^2}{a^2 + ab} \\ & \stackrel{\text{Bergstrom}}{\leq} \sqrt{\frac{a^2 + b^2 + c^2}{ab + bc + ca}} \cdot \frac{(a+b+c)^2}{a^2 + b^2 + c^2 + ab + bc + ca} = \frac{x+2y}{x+y} \sqrt{\frac{x}{y}} \end{aligned}$$

Where  $x = a^2 + b^2 + c^2$ ,  $y = ab + bc + ca$  and  $x \geq y$

$$\begin{aligned} & \text{We need to prove: } \frac{x+2y}{x+y} \sqrt{\frac{x}{y}} \geq \frac{3}{2} \Leftrightarrow 4x(x+2y)^2 \geq 9y(x+y)^2 \Leftrightarrow \\ & 4x(x^2 + 4xy + 4y^2) \geq 9y(x^2 + 2xy + y^2) \Leftrightarrow 4x^3 + 7x^2y - 2xy^2 - 9y^3 \geq 0 \Leftrightarrow \\ & (x-y)(4x^2 + 11xy + 9y^2) \geq 0, \text{ which is true } x-y \geq 0 \end{aligned}$$

$$\sum_{\text{cyc}} \frac{a}{a+b} \cdot \sqrt{\frac{a^2 + b^2 + c^2}{ab + bc + ca}} \geq \frac{3}{2}. \text{Proved.}$$

**613. If  $x_1, x_2, \dots, x_n > 0$  then:**

$$2 \sum_{i=1}^n \left( \frac{1}{x_i + x_{i+1}} \prod_{\substack{j=1 \\ j \neq i}}^n x_j \right) \leq \prod_{\substack{j=1 \\ j \neq i}}^n x_j, \quad x_{n+1} = x_1$$

*Proposed by Marin Chirciu-Romania*



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**Solution by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned}
 2 \sum_{i=1}^n \left( \frac{1}{x_i + x_{i+1}} \prod_{\substack{j=1 \\ j \neq i}}^n x_j \right) &= 2 \left( \frac{x_2 x_3 \dots x_n}{x_1 + x_2} + \frac{x_1 x_3 \dots x_n}{x_2 + x_3} + \dots + \frac{x_1 x_2 \dots x_{n-1}}{x_n + x_1} \right) \\
 &= 2 x_1 x_2 \dots x_n \left( \frac{1}{x_1(x_1 + x_2)} + \frac{1}{x_2(x_2 + x_3)} + \frac{1}{x_n(x_n + x_1)} \right) = \Omega \\
 \frac{2}{2x_1(x_1 + x_2)} &\stackrel{CBS}{\geq} 2 \cdot \frac{1}{4} \cdot \left( \frac{1}{2x_1} + \frac{1}{x_1 + x_2} \right) \leq \frac{2}{4} \cdot \left( \frac{1}{2x_1} + \frac{1}{4} \left( \frac{1}{x_1 + x_2} \right) \right) = 2 \left( \frac{3}{16x_1} + \frac{1}{16x_2} \right)
 \end{aligned}$$

*and analogs.*

$$\begin{aligned}
 \Rightarrow \Omega &\leq 4 x_1 x_2 \dots x_n \left[ \frac{1}{16} \left( \left( \frac{3}{x_1} + \frac{1}{x_2} \right) + \left( \frac{3}{x_2} + \frac{1}{x_3} \right) + \dots + \left( \frac{3}{x_n} + \frac{1}{x_1} \right) \right) \right] \\
 &= 4 x_1 x_2 \dots x_n \left( \frac{1}{16} \sum_{i=1}^n \frac{4}{x_i} \right) = 4 \cdot \frac{4}{16} x_1 x_2 \dots x_n \sum_{i=1}^n \frac{1}{x_i} = \sum_{i=1}^n \left( \prod_{\substack{j=1 \\ j \neq i}}^n x_j \right)
 \end{aligned}$$

**614. If  $x, y, z, t > 0$ ,  $xyzt = 1$  then:**

$$\frac{x^2 + 1}{x^5 + 3} + \frac{y^2 + 1}{y^5 + 3} + \frac{z^2 + 1}{z^5 + 3} + \frac{t^2 + 1}{t^5 + 3} \leq 2$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned}
 \frac{x^2 + 1}{x^5 + 3} &\leq \frac{2}{x + 3} \quad (*) \\
 \Leftrightarrow 2(x^5 + 3) - (x + 3)(x^2 + 1) &\geq 0 \\
 2x^5 - x^3 - 3x^2 - x + 3 &\geq 0 \\
 (x - 1)^2(x + 1)(2x^2 + 2x + 3) &\geq 0 \text{ true for } x \geq 0 \Rightarrow (*) \text{ is true.}
 \end{aligned}$$

$$\begin{aligned}
 LHS &= \sum_{cyc} \frac{x^2 + 1}{x^5 + 3} \leq \sum_{cyc} \frac{2}{x + 3} = 2 \sum_{cyc} \frac{1}{x + 3} = 2 \left( \frac{1}{x + 3} + \frac{1}{y + 3} + \frac{1}{z + 3} + \frac{1}{t + 3} \right) = 2\Phi \\
 \Phi &= \frac{1}{x + 3} + \frac{1}{y + 3} + \frac{1}{z + 3} + \frac{1}{t + 3} \leq 1 \Leftrightarrow
 \end{aligned}$$



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$$\sum_{cyc} (x+3)(y+3)(z+3) \leq (x+3)(y+3)(z+3)(t+3) \Leftrightarrow$$

$$xyzt + 2(xyz + xyt + xzt + yzt) + 3(xy + xz + xt + yz + yt + zt) \geq 27 \stackrel{xyzt=1}{\Longleftrightarrow}$$

$$2(xyz + xyt + xzt + yzt) + 3(xy + xz + xt + yz + yt + zt) \geq 26 \quad (**)$$

$$2(xyz + xyt + xzt + yzt) \stackrel{Am-Gm}{\geq} 2 \cdot 4\sqrt[4]{(xyz)^3} = 8 \quad (1)$$

$$3(xy + xz + xt + yz + yt + zt) \stackrel{Am-Gm}{\geq} 3 \cdot 6\sqrt[6]{(xyz)^3} = 18 \quad (2)$$

$\stackrel{(1)+(2)}{\Longrightarrow} (**)$  is true. Proved. Equality  $\Leftrightarrow x = y = z = t = 1$

**615. If  $a, b, c > 0, abc = 1$  then:**

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{3(a + b + c)}{(ab + bc + ca)^2} \geq 2$$

*Proposed by George Apostolopoulos—Messolonghi-Greece*

**Solution 1 by Adrian Popa-Romania**

$$\begin{aligned} \frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{3(a + b + c)}{(ab + bc + ca)^2} &\geq 2 \Leftrightarrow \\ (a^2 + b^2 + c^2)(ab + bc + ca) + 3abc &\geq 2(ab + bc + ca)^2 \Leftrightarrow \\ a^3b + b^3c + c^3a + a^3c + b^3a + c^3b &\geq 2(a^2b^2 + b^2c^2 + c^2a^2) \Leftrightarrow \\ \frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} &\geq \frac{2}{c^2} + \frac{2}{b^2} + \frac{2}{a^2} \\ \frac{a^2 + b^2}{c} + \frac{b^2 + c^2}{a} + \frac{a^2 + c^2}{b} &\stackrel{Am-Gm}{\geq} \frac{2ab}{c} + \frac{2bc}{a} + \frac{2ca}{b} = \frac{2}{c^2} + \frac{2}{b^2} + \frac{2}{a^2} \end{aligned}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

For  $a, b, c > 0$  we have:

$$ab(a^2 + b^2) + bc(b^2 + c^2) + ac(a^2 + c^2) \stackrel{Am-Gm}{\geq} 2(a^2b^2 + b^2c^2 + c^2a^2) \quad (*)$$

Let:  $p = a + b + c; q = ab + bc + ca; r = abc$

$$(*) \Leftrightarrow p^2q + 3pr \geq 4q^2$$

Because:  $abc = 1$ , we have



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$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{3(a+b+c)}{(ab+bc+ca)^2} \geq 2 \Leftrightarrow$$

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{3abc(a+b+c)}{(ab+bc+ca)^2} \geq 2 \Leftrightarrow$$

$$(a^2+b^2+c^2)(ab+bc+ca) + 3abc(a+b+c) \geq 2(ab+bc+ca)^2 \Leftrightarrow$$

$$(p^2 - 2q)q + 3rp \geq 2q^2 \Leftrightarrow p^2q + 3pr \geq 4q^2 \text{ true by (*)}$$

*Proved. Equality  $\Leftrightarrow a = b = c = 1$*

**Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $a, b, c > 0, abc = 1$  we have*

$$\begin{aligned} & \frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{3(a+b+c)}{(ab+bc+ca)^2} = \\ &= \frac{a^3b + b^3c + c^3a + a^3c + b^3a + c^3b + a^2bc + ab^2c + abc^2 + 2(a+b+c)}{a^2b^2 + b^2c^2 + c^2a^2 + 2(ab^2c + ab^2c + abc^2)} \\ &\geq \frac{2(a^2b^2 + b^2c^2 + c^2a^2) + 4(a+b+c)}{a^2b^2 + b^2c^2 + c^2a^2 + 2(a+b+c)} = 2 \end{aligned}$$

**616. If  $a, b, c > 0, ab + bc + ca \geq 3$  then:**

$$\left(a^3 + \frac{2}{3}\right)\left(b^3 + \frac{2}{3}\right)\left(c^3 + \frac{2}{3}\right) \geq \frac{125}{27}$$

*Proposed by George Apostolopoulos-Messolonghi-Greece*

**Solution 1 by Michael Sterghiou-Greece**

$$\left(a^3 + \frac{2}{3}\right)\left(b^3 + \frac{2}{3}\right)\left(c^3 + \frac{2}{3}\right) \geq \frac{125}{27} \quad (1)$$

*Let:  $(p, q, r) = (\sum a, \sum ab, \prod a)$ . As  $q \geq 3$  then  $p \geq 3$ .*

*Assume  $r \geq 1$  then*

$$\begin{aligned} LHS &= \prod \left( \frac{2a^3}{3} + \frac{a^3}{3} + \frac{1}{3} + \frac{1}{3} \right) \stackrel{Am-Gm}{\geq} \prod \left( \frac{2a^3}{3} + a \right) = r \cdot \prod \left( \frac{2a^2}{3} + 1 \right) \\ &= r \cdot \prod \left[ \left( \frac{a^2}{3} + \frac{a^2}{3} + \frac{1}{3} + \frac{1}{3} \right) + \frac{1}{3} \right] \stackrel{Am-Gm}{\geq} r \cdot \prod \frac{4a+1}{3} \text{ which it's suffices to be } \geq \frac{125}{27} \text{ or} \\ &r \cdot \prod (4a+1) \geq 125 \text{ or } 64r + 4p + 16q + 1 \geq 125 \text{ which holds as } r \geq 1, p \geq 3, \\ &q \geq 3. \text{ Assume now } r \leq 1. \end{aligned}$$



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$$\text{Expanding (1) we get: } r^3 + \frac{2}{3} \sum a^3 b^3 + \frac{4}{9} \sum a^3 + \frac{8}{27} \geq \frac{125}{27} \quad (2)$$

$$\text{But: } \sum a^3 b^3 = q^3 - 3pqr + 3r^2$$

$$\sum a^3 = p^3 - 3pq + 3r$$

$$\text{So, (2) becomes: } 4p^3 - 18pqr - 12pq + 6q^3 + 9r^3 + 18r^2 + 12r - 39 = f(r)$$

$$f'(r) = 3[(3r+2)^2 - 6pq] < 0 \text{ as } (3r+2)^2 < 25 \text{ and } 6pq \geq 54 \quad (r \leq 1) \text{ hence } f \downarrow$$

$$\text{and } f(r) \geq f(1), r \leq 1$$

$$\begin{aligned} f(1) &= 4p^3 + 6q^3 - 30pq = 2[2p^3 + 3q^3 - 15pq] \\ &= 2[2(p^3 + q^3 + 27) - 54 + q^3 - 15pq] \geq \\ &2[18pq + q^3 - 15pq - 54] \geq 2[q^3 - 3pq - 54] \geq 0 \end{aligned}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $a, b, c > 0$  and  $ab + bc + ca \geq 3$  we have

$$\begin{aligned} (2a^3 + 1)(2b^3 + 1)(2c^3 + 1) &= (a^3 + 1)(b^3 + 1)(c^3 + 1) \\ &\geq (ab + bc + ca)^3 \geq 27: (ab + bc + ca) \geq 3 \end{aligned}$$

$$\text{Hence } \sqrt[3]{(2a^3 + 1)(2b^3 + 1)(2c^3 + 1)} \geq 3 \text{ and}$$

$$\begin{aligned} &(a^3 + 1)(b^3 + 1)(c^3 + 1)(c^3 + 1)(a^3 + 1) \\ &= (a^3 b^3 + a^3 + b^3 + 1)(b^3 c^3 + b^3 + c^3 + 1)(c^3 a^3 + c^3 + a^3 + 1) \\ &\geq ((ab)^2 + (bc)^2 + (ca)^2)^3 \geq (ab + bc + ca)^3: (ab + bc + ca) \geq 64 \end{aligned}$$

$$\text{Hence: } (a^3 + 1)(b^3 + 1)(c^3 + 1) \geq 8$$

$$\sqrt[3]{(a^3 + 1)(b^3 + 1)(c^3 + 1)} \geq 2$$

$$\sqrt[3]{(a^3 + 1)(b^3 + 1)(c^3 + 1)} + \sqrt[3]{(2a^3 + 1)(2b^3 + 1)(2c^3 + 1)} \geq 5$$

$$\left( \sqrt[3]{(a^3 + 1)(b^3 + 1)(c^3 + 1)} + \sqrt[3]{(2a^3 + 1)(2b^3 + 1)(2c^3 + 1)} \right)^3 \geq 125$$

$$(a^3 + 1) + (2a^3 + 1) \quad (b^3 + 1) + (2b^3 + 1) \quad (c^3 + 1) + (2c^3 + 1) \geq 125$$

$$(3a^3 + 2)(3b^3 + 2)(3c^3 + 2) \geq 125$$

$$\frac{(3a^3 + 2)(3b^3 + 2)(3c^3 + 2)}{27} \geq \frac{125}{27}$$

$$\left(a^3 + \frac{2}{3}\right) \left(b^3 + \frac{2}{3}\right) \left(c^3 + \frac{2}{3}\right) \geq \frac{125}{27}$$



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**617. If  $a, b, c > 0$  then:**

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(9 + a^2b^2 + b^2c^2 + c^2a^2) \geq 36$$

*Proposed by Nguyen Van Canh-Ben Tre-Vietnam*

**Solution 1 by Rahim Shahbazov-Baku-Azerbaijan**

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(9 + a^2b^2 + b^2c^2 + c^2a^2) \geq 36 \quad (1)$$

$$x^2 + 1 \geq 2x, \forall x > 0$$

$$\begin{aligned} 9 + a^2b^2 + b^2c^2 + c^2a^2 &= (a^2b^2 + 1) + (b^2c^2 + 1) + (c^2a^2 + 1) + 6 \\ &\geq 2(ab + bc + ca + 3) \stackrel{(1)}{\Rightarrow} \end{aligned}$$

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(ab + bc + ca + 3) \geq 18 \quad (2)$$

$$\begin{aligned} LHS &= 2(a + b + c) + \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} + 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \\ &\geq 3(a + b + c) + 3 \cdot \frac{9}{a + b + c} \geq 18 \stackrel{x=a+b+c}{=} x + \frac{9}{x} \geq 6 \Leftrightarrow (x - 3)^2 \geq 0. \end{aligned}$$

*Equality for  $a = b = c = 1$ . Proved.*

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

$$\begin{aligned} &\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(9 + a^2b^2 + b^2c^2 + c^2a^2) \\ &= 9\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + ab^2 + ac^2 + \frac{b^2c^2}{a} + a^2b + bc^2 + \frac{c^2a^2}{b} + a^2c + b^2c + \frac{a^2b^2}{c} \\ &= \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + ab^2\right) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + a^2c\right) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + a^2b\right) \\ &\quad + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + bc^2\right) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + b^2c\right) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + ac^2\right) \\ &\quad + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{b^2c^2}{a}\right) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{c^2a^2}{b}\right) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{a^2b^2}{c}\right) \\ &\geq 6 \cdot 4 + 4 \left( \sqrt[4]{\frac{a}{b}} + \sqrt[4]{\frac{b}{c}} + \sqrt[4]{\frac{c}{a}} \right) = 36 \end{aligned}$$

*Equality for  $a = b = c = 1$ .*



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**Solution 3 by Marian Ursărescu-Romania**

*From  $x^2 + y^2 + z^2 \geq xy + yz + zx$  we have:*

$$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c) \geq 3abc\sqrt[3]{abc} \Rightarrow$$

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 3\sqrt[3]{a^4b^4c^4}$$

$$\text{We must show that: } \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(9 + 3\sqrt[3]{a^4b^4c^4}) \geq 36$$

$$(ab + bc + ca)\left(3 + \sqrt[3]{a^4b^4c^4}\right) \geq 12abc \quad (1)$$

$$\text{But: } ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} \quad (2)$$

$$3 + \sqrt[3]{a^4b^4c^4} = 1 + 1 + 1 + \sqrt[3]{a^4b^4c^4} \geq 4\sqrt[4]{\sqrt[3]{a^4b^4c^4}} = 4\sqrt[3]{abc} \quad (3)$$

$$\text{From (1)+(2) we have: } (ab + bc + ca)\left(3 + \sqrt[3]{a^4b^4c^4}\right) \geq 12\sqrt[3]{a^3b^3c^3} = 12abc$$

**Solution 4 by Michael Sterghiou-Greece**

*Let:  $(p, q, r) = (\sum a, \sum ab, \prod a)$ . We have:*

$$\sum \frac{1}{a} = \frac{q}{r}; \sum a^2b^2 = q^2 - 2pr$$

*So, (1) reduces to:  $q^3 + 9q - 2pqr - 36r \geq 0$ .*

*As  $q^2 \geq 3pr$  it is enough that:*

$$q \cdot 3pr + 9q - 2pqr - 36r \geq 0 \text{ or } 9q - pqr - 36 \geq 0 \quad (2)$$

*But:  $q \geq 3\sqrt[3]{r^2}, p \geq 3\sqrt[3]{r}$  so (2) reduces to:*

$$9\sqrt[3]{r^2}(\sqrt[3]{r^4} - 4\sqrt[3]{r} + 3) \geq 0 \text{ or } 9\sqrt[3]{r^2}(\sqrt[3]{r} - 1)^2(\sqrt[3]{r^2} + 2\sqrt[3]{r} + 3) \geq 0 \text{ which holds. Done!}$$

**618. Let  $x, y, z > 0$ . Prove:**

$$x^3 + y^3 + z^3 - (x + y + z) \geq 2\log(xyz)$$

*Proposed by Jalil Hajimir-Toronto-Canada*

**Solution 1 by Daniel Sitaru-Romania**

$$f: (0, \infty) \rightarrow \mathbb{R}, f(x) = x^3 - x - 2\log x$$

$$f'(x) = 3x^2 - 1 - \frac{2}{x} = \frac{(x-1)(3x^2+3x+2)}{x}$$



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$$sgn f'(x) = sgn(x - 1), \min f(x) = f(1) = 0 \rightarrow f(x) \geq 0$$

$$\begin{cases} x^3 - x - 2\log x \geq 0 \\ y^3 - y - 2\log y \geq 0 \\ z^3 - z - 2\log z \geq 0 \end{cases} \rightarrow \begin{cases} x^3 - x \geq 2\log x \\ y^3 - y \geq 2\log y \\ z^3 - z \geq 2\log z \end{cases} \rightarrow$$

$$\rightarrow x^3 + y^3 + z^3 - (x + y + z) \geq 2\log x + 2\log y + 2\log z = 2\log(xyz)$$

*Equality holds for  $x = y = z = 1$ .*

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For all  $a > 0$ , we have  $a^3 - a + 1 \geq a^2 + 1 \Leftrightarrow a^3 - a^2 - a + 1 \geq 0$

$$a^2(a - 1) - (a - 1) = (a^2 - 1)(a - 1) = (a - 1)^2(a + 1) \geq 0$$

$$\text{Hence } e^{a^3-a} \geq a^2 \Leftrightarrow e^{a^3} \geq e^a \cdot a^2 \Leftrightarrow \log(e^{a^3}) \geq \log(e^a \cdot a^2)$$

$$a^3 \geq a + \log a^2; t \rightarrow \log t \text{-increasing.}$$

Hence for  $x, y, z > 0$ , we have  $x^3 + y^3 + z^3 - (x + y + z) \geq 2\log(xyz)$

$$x^3 + y^3 + z^3 \geq 2\log(xyz) + x + y + z$$

If  $\log e^{x^3} + \log e^{y^3} + \log e^{z^3} \geq (\log x^2 + \log e^x) + (\log y^2 + \log e^y) + (\log z^2 + \log e^z) = \log(e^x \cdot x^2) + \log(e^y \cdot y^2) + \log(e^z \cdot z^2)$  and is true,

$$\text{because: } e^{x^3} \geq e^x \cdot x^2; e^{y^3} \geq e^y \cdot y^2; e^{z^3} \geq e^z \cdot z^2$$

**619. If  $a, b, c, n > 0$ ,  $(a^2 - na + n^2)(b^2 - nb + n^2)(c^2 - nc + n^2) = 1$**

**then:**

$$a^2b^2 + b^2c^2 + c^2a^2 + n^4 \leq \frac{4}{n^2}$$

*Proposed by Marin Chirciu, Octavian Stroe-Romania*

**Solution by Șerban Florin George-Romania**

$$(x - 1)^4 \geq 0, \forall x \in \mathbb{R} \Leftrightarrow x^4 - 4x^3 + 6x^2 - 4x + 1 \geq 0 \Leftrightarrow$$

$$(x^4 - 2x^3 + 3x^2 - 2x + 1) + (-2x^3 + 3x^2 - 2x) \geq 0$$

$$(x^2 - x + 1)^2 + (x^4 - 2x^3 + 3x^2 - 2x + 1) \geq x^4 + 1$$

$$2(x^2 - x + 1)^2 \geq x^4 + 1 \Rightarrow \prod_{cyc} (a^2 - na + n^2) = 1 \Rightarrow \prod_{cyc} \left( \left(\frac{a}{n}\right)^2 - \frac{a}{n} + 1 \right) = \frac{1}{n^6}$$



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$$\text{Denote: } \frac{a}{n} = x; \frac{b}{n} = y; \frac{c}{z} = z \Rightarrow \prod_{cyc} (x^2 - x + 1)^2 = \frac{1}{n^{12}}$$

$$\begin{aligned} \prod_{cyc} (x^2 - x + 1)^2 &= \frac{1}{n^{12}} \geq \frac{(x^4 + 1)(y^4 + 1)(z^4 + 1)}{2 \cdot 2 \cdot 2} = \frac{(2x^4 + 2)(x^4y^4 + x^4 + y^4 + 1)}{16} \\ &= \frac{(x^4 + x^4 + 1 + 1)(x^4y^4 + x^4 + y^4 + 1)}{16} \\ &= \frac{((x^2)^2 + (x^2)^2 + 1^2 + 1^2)((x^2)^2 + (y^2)^2 + (x^2)^2(y^2)^2 + 1^2)}{16} \\ &\stackrel{C.B.S.}{\geq} \frac{(x^2 + y^2 + x^2y^2 + 1)^2}{16} \Rightarrow \frac{1}{n^6} \geq \frac{\sum_{cyc} x^2y^2 + 1}{4} \\ &\frac{4}{n^6} \geq \sum_{cyc} \frac{a^2b^2}{n^4} + 1 \Rightarrow \frac{4}{n^2} \geq \sum_{cyc} a^2b^2 + n^4 \end{aligned}$$

**620. If  $x, y, z \geq 0, x + y + z = 1$  then:**

$$\mu(x^3 + y^3 + z^3) + 1 \geq \left(3 + \frac{\mu}{3}\right)(x^2 + y^2 + z^2), \mu \geq 6$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

*Because:  $x + y + z = 1$*

$$\mu(x^3 + y^3 + z^3) + 1 \geq \left(3 + \frac{\mu}{3}\right)(x^2 + y^2 + z^2) \Leftrightarrow$$

$$3\mu(x^3 + y^3 + z^3) + 3(x + y + z)^3 \stackrel{(*)}{\geq} (9 + \mu)(x^2 + y^2 + z^2)(x + y + z)$$

*Let:  $p = x + y + z; q = xy + yz + zx; r = xyz$*

$$\begin{aligned} (*) &\Leftrightarrow 3\mu(p^3 - 3pq + 3r) + 3p^3 \geq (9 + \mu)(p^2 - 2q)p \Leftrightarrow \\ &2(\mu - 3)p^3 + (18 - 7\mu)pq + 9\mu r \geq 0 \end{aligned}$$

$$r \stackrel{\text{Schur's}}{\geq} \frac{p(4q - p^2)}{9} \Rightarrow 9r \geq 4pq - p^3 \Rightarrow 9\mu r \geq 4\mu pq - \mu p^3$$

*We need to prove:*

$$\begin{aligned} 2(\mu - 3)p^3 + (18 - 7\mu) + 4\mu pq - \mu p^3 &\geq 0 \Leftrightarrow (\mu - 6)p^3 + 3(6 - \mu)pq \geq 0 \\ &\Leftrightarrow (\mu - 6) \cdot p \cdot (p^2 - 3q) \geq 0 \end{aligned}$$

*Which is true because:  $\mu \geq 6 \Rightarrow \mu - 6 \geq 0$*



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$$(a+b+c)^2 \geq 3(ab+bc+ca) \Rightarrow p^2 \geq 3q. \text{ Proved.}$$

**621. If  $x, y, z \geq 2$  then:**

$$\sum_{cyc} \frac{1}{x+1} = 1 \Rightarrow \sum_{cyc} \frac{3x^2 + x + 4}{(x+1)(x^4+2)} + 2 \leq 2 \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

For  $x, y, z \geq 2$  and  $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$ . We may write Inequality as:

$$\begin{aligned} & \frac{3x^2 + x + 4}{(x+1)(x^4+2)} + \frac{3y^2 + y + 4}{(y+1)(y^4+2)} + \frac{3z^2 + z + 4}{(z+1)(z^4+2)} + \\ & + 2 \left( \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \right) \leq 2 \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right); \\ \Leftrightarrow & \sum \left( \frac{3x^2 + x + 4}{(x+1)(x^4+2)} + \frac{2}{x+1} \right) \leq 2 \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right); \\ \Leftrightarrow & \sum \left( \frac{2x^4 + 3x^2 + x + 8}{(x+1)(x^4+2)} \right) \leq \sum \frac{2}{x}; \quad (*) \end{aligned}$$

Hence, we must show that:

$$\begin{aligned} & \frac{2x^4 + 3x^2 + x + 8}{(x+1)(x^4+2)} \leq \frac{2}{x}; \quad (\forall x \geq 2) \\ \Leftrightarrow & x(2x^4 + 3x^2 + x + 8) \leq 2(x+1)(x^4+2) \\ \Leftrightarrow & 2x^4 - 3x^3 - x^2 - 4x + 4 \geq 0 \Leftrightarrow (x-2)(2x^3 + x^2 + x - 2) \geq 0 \end{aligned}$$

Which is true because:

$$x \geq 2 \rightarrow x-2 \geq 0; 2x^3 + x^2 + x - 2 \geq 16 + 4 + 2 - 2 = 20 > 0$$

Similary:

$$\begin{aligned} & \frac{2y^4 + 3y^2 + y + 8}{(y+1)(y^4+2)} \leq \frac{2}{y}; \quad (\forall y \geq 2) \\ \Leftrightarrow & \frac{2z^4 + 3z^2 + z + 8}{(z+1)(z^4+2)} \leq \frac{2}{z}; \quad (\forall z \geq 2) \\ \rightarrow & (*) \text{ is true. Proved.} \end{aligned}$$



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**622. If  $a, b, c > 0$  then:**

$$\frac{(\sum_{cyc} ab) \left( \sum_{cyc} \frac{1}{ab} \right)}{(\sum_{cyc} \sqrt[3]{a})(\sum_{cyc} \sqrt[3]{a^2})} \geq \frac{\left( \sum_{cyc} \frac{1}{\sqrt[3]{a}} \right) \left( \sum_{cyc} \frac{1}{\sqrt[3]{a^2}} \right)}{(\sum_{cyc} a^2 b^2) \left( \sum_{cyc} \frac{1}{a^2 b^2} \right)}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $a, b, c > 0$ , we give  $a = x^3, b = y^3, c = z^3$

$$\frac{(\sum_{cyc} (xy)^3) \left( \sum_{cyc} \frac{1}{(xy)^3} \right)}{(\sum_{cyc} x)(\sum_{cyc} x^2)} \geq \frac{\left( \sum_{cyc} \frac{1}{x} \right) \left( \sum_{cyc} \frac{1}{x^2} \right)}{(\sum_{cyc} (xy)^6) \left( \sum_{cyc} \frac{1}{(xy)^6} \right)}$$

$$(\sum_{cyc} (xy)^3) \left( \sum_{cyc} \frac{1}{(xy)^3} \right) (\sum_{cyc} (xy)^6) \left( \sum_{cyc} \frac{1}{(xy)^6} \right) \geq (\sum_{cyc} x)(\sum_{cyc} x^2) \left( \sum_{cyc} \frac{1}{x} \right) \left( \sum_{cyc} \frac{1}{x^2} \right)$$

*Hence*

$$(\sum_{cyc} (xy)^3) \frac{(\sum_{cyc} x^3)}{\prod_{cyc} x^3} (\sum_{cyc} (xy)^6) \frac{(\sum_{cyc} x^6)}{\prod_{cyc} x^6} \geq (\sum_{cyc} x)(\sum_{cyc} x^2) \frac{(\sum_{cyc} xy)}{\prod_{cyc} x} \cdot \frac{(\sum_{cyc} (xy)^2)}{\prod_{cyc} (xy)^2}$$

*Hence*

$$(\sum_{cyc} (xy)^3) (\sum_{cyc} x^3) (\sum_{cyc} (xy)^6) (\sum_{cyc} x^6) \geq (\sum_{cyc} x)(\sum_{cyc} x^2) (\sum_{cyc} xy) (\sum_{cyc} (xy)^2)$$

*Hence*

$$\begin{aligned} & \frac{(\sum_{cyc} xy)(\sum_{cyc} (xy)^2)}{3} \cdot \frac{(\sum_{cyc} x)(\sum_{cyc} x^2)}{3} \cdot \left( 3 \prod_{cyc} x^4 \right) \left( 3 \prod_{cyc} x^2 \right) \geq \\ & \geq \left( \sum_{cyc} xy \right) \left( \sum_{cyc} (xy)^2 \right) + \left( \sum_{cyc} x \right) \left( \sum_{cyc} x^2 \right) \left( \prod_{cyc} x^6 \right) \end{aligned}$$

*therefore is true, because*

$$3 \sum_{cyc} \frac{1}{x^3} \geq 3 \sum_{cyc} \frac{1}{x^2 z} \Rightarrow \sum_{cyc} \frac{1}{x^3} \geq \sum_{cyc} \frac{1}{x^2 z} \Rightarrow \sum_{cyc} (xy)^3 \geq \sum_{cyc} xy^3 z^2$$

*And*

$$3 \sum_{cyc} \frac{1}{x^3} \geq 3 \sum_{cyc} \frac{1}{x^2 y} \Rightarrow \sum_{cyc} \frac{1}{x^3} \geq \sum_{cyc} \frac{1}{x^2 y} \Rightarrow \sum_{cyc} (xy)^3 \geq \sum_{cyc} xy^2 z^3$$



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*Similarly*

$$\sum_{cyc} x^3 \geq \sum_{cyc} x^2 y \text{ and } \sum_{cyc} x^3 \geq \sum_{cyc} x^2 z$$

**Solution 2 by Sudhir Jha-Kolkata-India**

$$\frac{\left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{1}{ab}\right)}{\left(\sum_{cyc} \sqrt[3]{a}\right)\left(\sum_{cyc} \sqrt[3]{a^2}\right)} \geq \frac{\left(\sum_{cyc} \frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc} \frac{1}{\sqrt[3]{a^2}}\right)}{\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} \frac{1}{a^2 b^2}\right)}; \quad (1)$$

*Hence*

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{1}{ab}\right)\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} \frac{1}{a^2 b^2}\right) \geq \left(\sum_{cyc} \sqrt[3]{a}\right)\left(\sum_{cyc} \sqrt[3]{a^2}\right)\left(\sum_{cyc} \frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc} \frac{1}{\sqrt[3]{a^2}}\right)$$

*Hence*

$$\begin{aligned} & \left(\sum_{cyc} ab\right)\left(\sum_{cyc} \frac{c}{abc}\right)\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} \frac{c^2}{a^2 b^2 c^2}\right) \geq \\ & \geq \left(\sum_{cyc} \sqrt[3]{a}\right)\left(\sum_{cyc} \sqrt[3]{a^2}\right)\left(\sum_{cyc} \frac{\sqrt[3]{bc}}{\sqrt[3]{abc}}\right)\left(\sum_{cyc} \frac{\sqrt[3]{b^2 c^2}}{\sqrt[3]{a^2 b^2 c^2}}\right) \end{aligned}$$

*Hence*

$$(\sum_{cyc} ab)(\sum_{cyc} a)(\sum_{cyc} a^2 b^2)(\sum_{cyc} a^2) \geq a^2 b^2 c^2 (\sum_{cyc} \sqrt[3]{a})(\sum_{cyc} \sqrt[3]{a^2})(\sum_{cyc} \sqrt[3]{ab})(\sum_{cyc} \sqrt[3]{a^2 b^2}); \quad (2)$$

*By Chebyshev's inequality, we have:*

$$\sum_{cyc} ab \geq \frac{(\sum_{cyc} \sqrt[3]{ab})(\sum_{cyc} \sqrt[3]{a^2 b^2})}{3}; \quad (3)$$

$$\left(\sum_{cyc} a\right) \geq \frac{(\sum_{cyc} \sqrt[3]{a})(\sum_{cyc} \sqrt[3]{a^2})}{3}; \quad (4)$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \stackrel{Am-Gm}{\leq} 3 \cdot \sqrt[3]{\prod_{cyc} a^4} \cdot 3 \cdot \sqrt[3]{\prod_{cyc} a^2} = 9a^2 b^2 c^2; \quad (5)$$

*Multiplying (3),(4),(5) we get (2) is true, then (1) is true. Proved.*



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**623. If  $a, b, c > 2, a + b + c = 9$  then:**

$$\Gamma\left(\frac{a}{b}\right) + \Gamma\left(\frac{b}{c}\right) + \Gamma\left(\frac{c}{a}\right) \geq 6$$

*Proposed by Jalil Hajimir-Toronto-Canada*

**Solution by Daniel Sitaru-Romania**

$$\begin{aligned}
 \Gamma\left(\frac{a}{b}\right) + \Gamma\left(\frac{b}{c}\right) + \Gamma\left(\frac{c}{a}\right) &= \\
 &= \sum_{cyc} \left( \frac{2}{3} \Gamma\left(\frac{a}{b}\right) + \frac{1}{3} \Gamma\left(\frac{b}{c}\right) \right)^{JENSEN} \stackrel{\geq}{\approx} \sum_{cyc} \Gamma\left(\frac{\frac{2a}{b} + \frac{b}{c}}{3}\right) \geq \\
 &\stackrel{JENSEN}{\geq} 3 \Gamma\left(\sum_{cyc} \frac{\frac{2a}{b} + \frac{b}{c}}{3}\right) = 3 \Gamma\left(\frac{\frac{2a}{b} + \frac{b}{c} + \frac{2b}{c} + \frac{c}{a} + \frac{2c}{a} + \frac{a}{b}}{3}\right) = \\
 &= 3 \Gamma\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \stackrel{a,b,c>2}{\geq} 3 \Gamma\left(3 \sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}}\right) = \\
 &= 3 \Gamma(3) = 3 \cdot 2! = 6
 \end{aligned}$$

*Equality holds for  $a = b = c = 3$ .*

**624. If  $a, b, c > 0$  then:**

$$\frac{(a^2 - ab + b^2)^6}{(a+b)^{12}} + \frac{(b^2 - bc + c^2)^6}{(b+c)^{12}} + \frac{(c^2 - ca + a^2)^6}{(c+a)^{12}} \geq \frac{3}{4096}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Abner Chinga Bazo-Lima-Peru**

$$\begin{aligned}
 (a-b)^2 \geq 0 &\Leftrightarrow a^2 + b^2 \geq 2ab \Leftrightarrow 3(a^2 + b^2) \geq 6ab \\
 4(a^2 + b^2) - 4ab &\geq a^2 + b^2 + 2ab \Leftrightarrow 4(a^2 - ab + b^2) \geq (a+b)^2 \\
 &\Leftrightarrow \frac{a^2 - ab + b^2}{(a+b)^2} \geq \frac{1}{4} \Leftrightarrow \frac{(a^2 - ab + b^2)^6}{(a+b)^{12}} \geq \frac{1}{2^{12}} \\
 \frac{(a^2 - ab + b^2)^6}{(a+b)^{12}} + \frac{(b^2 - bc + c^2)^6}{(b+c)^{12}} + \frac{(c^2 - ca + a^2)^6}{(c+a)^{12}} &\geq \frac{3}{4096}
 \end{aligned}$$



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**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $x, y > 0$ , we get that:*

$$\begin{aligned} \frac{4^6 \cdot (x^2 - xy + y^2)^6}{(x+y)^6} &= \frac{(4(x^2 - xy + y^2))^6}{(x+y)^6} = \\ &= \frac{(x^2 + x^2 + x^2 + x^2 + y^2 + y^2 + y^2 + y^2 - xy - xy - xy - xy)^6}{(x+y)^6} \\ &\geq \frac{(x^2 + x^2 + y^2 + y^2)^6}{(x+y)^{12}} \geq \frac{\left(\frac{(x+x+y+y)^2}{4}\right)^6}{(x+y)^{12}} \geq \frac{(x+y)^{12}}{(x+y)^{12}} = 1 \end{aligned}$$

*Hence:*

$$\begin{aligned} \frac{4^6 \cdot (a^2 - ab + b^2)^6}{(a+b)^{12}} + \frac{4^6 \cdot (b^2 - bc + c^2)^6}{(b+c)^{12}} + \frac{4^6 \cdot (c^2 - ca + a^2)^6}{(c+a)^{12}} &\geq 3 \\ \frac{(a^2 - ab + b^2)^6}{(a+b)^{12}} + \frac{(b^2 - bc + c^2)^6}{(b+c)^{12}} + \frac{(c^2 - ca + a^2)^6}{(c+a)^{12}} &\geq \frac{3}{4^6} = \frac{3}{4096} \end{aligned}$$

**625. If  $a, b, c > 0, \mu \geq 0, a + b + c + \mu abc = 8\mu + 6$  then:**

$$\left(1 + \mu ab + \frac{b}{c}\right) \left(1 + \mu bc + \frac{c}{a}\right) \left(1 + \mu ca + \frac{a}{b}\right) \geq 8(1 + 2\mu)^3$$

*Proposed by Marin Chirciu, Daniel Văcaru-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned} a, b, c > 0, a + b + c + \mu abc &= 8\mu + 6 \Rightarrow \mu abc = 8\mu + 6 - (a + b + c) \\ \left(1 + \mu ab + \frac{b}{c}\right) \left(1 + \mu bc + \frac{c}{a}\right) \left(1 + \mu ca + \frac{a}{b}\right) &\geq 8(1 + 2\mu)^3 \\ \Leftrightarrow (b + c + \mu abc)(a + c + \mu abc)(a + b + \mu abc) &\geq 8abc(1 + 2\mu)^3 \\ \Leftrightarrow (8\mu + 6 - a)(8\mu + 6 - b)(8\mu + 6 - c) &\geq 8abc(1 + 2\mu)^3 \\ \stackrel{k=8\mu+6}{\iff} (k-a)(k-b)(k-c) &\geq 8abc(1 + 2\mu)^3 \\ \Leftrightarrow k^3 - (a+b+c)k^2 + (ab+bc+ca)k - abc &\geq 8abc(1 + 2\mu)^3 \\ \Leftrightarrow k^3 - (a+b+c)k^2 + (ab+bc+ca)k &\geq abc[8(1 + 2\mu)^3 + 1] \\ \Leftrightarrow k^3 + (\mu abc - 8\mu - 6)k^2 + (ab+bc+ca)k &\geq abc[8(1 + 2\mu)^3 + 1] \end{aligned}$$



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$$\begin{aligned}
 & \Leftrightarrow k^3 - (8\mu + 6)k^2 + (ab + bc + ca)k \geq abc[8(1 + 2\mu)^3 + 1 - \mu k^2] \\
 & \Leftrightarrow (8\mu + 6)^3 - (8\mu + 6)(8\mu + 6)^2 + (8\mu + 6)(ab + bc + ca) \geq \\
 & \quad \geq abc[8(1 + 2\mu)^3 + 1 - \mu(8\mu + 6)^2] \\
 & \Leftrightarrow 2(4\mu + 3)(ab + bc + ca) \geq abc(4\mu + 3) \\
 & \xrightarrow[4\mu+3>0]{\Leftrightarrow} 2(ab + bc + ca) \stackrel{(*)}{\geq} 3abc; \\
 & 8\mu + 6 = a + b + c + \mu abc \stackrel{Am-Gm}{\geq} 3\sqrt[3]{abc} + \mu abc \\
 & \Leftrightarrow \mu t^3 + 3t - (8\mu + 6) \leq 0 \Leftrightarrow \mu(t - 2(t^2 + 2t +) + 3(t - 2)) \leq 0 \\
 & \Leftrightarrow (t - 2)(\mu t^2 + 2\mu t + 4\mu + 3) \leq 0 \xrightarrow[t>0,\mu\geq 0]{} t \leq 2 \Leftrightarrow abc \leq 2 \\
 & So, 2(ab + bc + ca) \stackrel{Am-Gm}{\geq} 2 \cdot 3\sqrt[3]{(abc)^2} \stackrel{(**)}{\geq} 3abc \\
 & (**) \Leftrightarrow 2\sqrt[3]{(abc)^2} \geq abc \Leftrightarrow 8(abc)^2 \geq (abc)^3 \Leftrightarrow 2 \geq abc \text{ (true), then (*) is true. Proved.}
 \end{aligned}$$

**626. If  $a, b, c, d > 0, abcd = 1$  then:**

$$\sum_{cyc} \frac{1 + (a^3 + b^3 + c^3)d}{a + b + c} \geq \frac{4}{3} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned}
 & a, b, c, d > 0, abcd = 1 \Rightarrow \\
 & \frac{1 + (a^3 + b^3 + c^3)d}{a + b + c} = \frac{(a^3 + b^3 + c^3 + abc)}{(a + b + c)abc} \stackrel{(1)}{\geq} \frac{4}{9} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\
 & (1) \Leftrightarrow 9(a^3 + b^3 + c^3 + abc) \geq 4(a + b + c)(ab + bc + ca) \\
 & \Leftrightarrow 9(a^3 + b^3 + c^3) + 9abc \stackrel{(*)}{\geq} 4(a + b + c)(ab + bc + ca)
 \end{aligned}$$

*By Schur's Inequality:*

$$a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(c + b) + ca(c + a)$$

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca)$$

$$But: a^3 + b^3 + c^3 \geq \frac{(a+b+c)^3}{3^2} \Leftrightarrow 9(a^3 + b^3 + c^3) \geq (a + b + c)^3$$



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$$\Leftrightarrow 9(a^3 + b^3 + c^3) + 9abc \geq (a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca) \Rightarrow$$

(\*) is true  $\Rightarrow$  (1) is true.

*Similarly:*

$$\frac{1 + (b^3 + c^3 + d^3)a}{b + c + d} \geq \frac{4}{9} \left( \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

$$\frac{1 + (c^3 + d^3 + a^3)d}{c + d + a} \geq \frac{4}{9} \left( \frac{1}{c} + \frac{1}{d} + \frac{1}{a} \right)$$

$$\frac{1 + (a^3 + b^3 + d^3)d}{a + b + d} \geq \frac{4}{9} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{d} \right)$$

$$\underset{\text{cyc}}{\Rightarrow} \sum \frac{1 + (a^3 + b^3 + c^3)d}{a + b + c} \geq \frac{4}{9} \left( \frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{3}{d} \right) = \frac{4}{3} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

**627. If  $a, b, c > 0$  then:**

$$\frac{4}{ac + b\sqrt{a}} + \frac{4}{ab + c\sqrt{b}} + \frac{4}{bc + a\sqrt{c}} \leq \frac{1 + \sqrt{a}}{bc} + \frac{1 + \sqrt{b}}{ca} + \frac{1 + \sqrt{c}}{ab}$$

*Proposed by Florică Anastase-Romania*

**Solution 1 by Adrian Popa-Romania**

Denote:  $\sqrt{a} = x, \sqrt{b} = y, \sqrt{c} = z \Rightarrow$

$$\frac{4}{x^2z^2 + y^2x} + \frac{4}{x^2y^2 + zy^2} + \frac{4}{y^2z^2 + x^2z} \leq \frac{1+x}{y^2z^2} + \frac{1+y}{x^2z^2} + \frac{1+z}{x^2y^2}$$

$$x^2z^2 + y^2x \stackrel{AGM}{\geq} 2xyz\sqrt{x}$$

$$x^2y^2 + zy^2 \stackrel{AGM}{\geq} 2xyz\sqrt{y}$$

$$y^2z^2 + x^2z \stackrel{AGM}{\geq} 2xyz\sqrt{z}$$

$$LHS \leq \frac{2}{2xyz\sqrt{x}} + \frac{2}{2xyz\sqrt{y}} + \frac{2}{2xyz\sqrt{z}} \stackrel{?}{\leq} \frac{x^2 + y^2 + z^2 + x^3 + y^3 + z^3}{(xyz)^2} \stackrel{\cdot ayxz}{\iff}$$

$$x^2 + x^3 \stackrel{AGM}{\geq} 2x^2\sqrt{x}$$

$$y^2 + y^3 \stackrel{AGM}{\geq} 2y^2\sqrt{y}$$



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$$z^2 + z^3 \stackrel{AGM}{\geq} 2z^2\sqrt{z}$$

**We must show that:**  $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} \leq \frac{2x^2\sqrt{x}}{xyz} + \frac{2y^2\sqrt{y}}{xyz} + \frac{2z^2\sqrt{z}}{xyz}$

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} \leq \frac{2x^2}{yz\sqrt{x}} + \frac{2y^2}{xz\sqrt{y}} + \frac{2z^2}{xy\sqrt{z}}$$

$$\frac{yz}{yz\sqrt{x}} + \frac{zx}{zx\sqrt{y}} + \frac{xy}{xy\sqrt{z}} \leq \frac{2x^2}{yz\sqrt{x}} + \frac{2y^2}{xz\sqrt{y}} + \frac{2z^2}{xy\sqrt{z}}$$

**Suppose:**  $x \geq y \geq z \Rightarrow \begin{cases} x^2 \geq y^2 \geq z^2 \\ xy\sqrt{z} \geq xz\sqrt{y} \geq yz\sqrt{x} \Rightarrow \frac{1}{yz\sqrt{x}} \leq \frac{1}{zx\sqrt{y}} \leq \frac{1}{xy\sqrt{z}} \\ xy \geq xz \geq yz \end{cases}$

*Applying Chebishev's inequality:*

$$\begin{aligned} & 3 \left( x^2 \cdot \frac{1}{yz\sqrt{x}} + y^2 \cdot \frac{1}{zx\sqrt{y}} + z^2 \cdot \frac{1}{xy\sqrt{z}} \right) \geq \\ & \geq (x^2 + y^2 + z^2) \left( \frac{1}{yz\sqrt{x}} + \frac{1}{zx\sqrt{y}} + \frac{1}{xy\sqrt{z}} \right) \geq \\ & \geq (yz + xz + xy) \left( \frac{1}{yz\sqrt{x}} + \frac{1}{zx\sqrt{y}} + \frac{1}{xy\sqrt{z}} \right) \geq \\ & \geq 3 \left( zy \cdot \frac{1}{yz\sqrt{x}} + xz \cdot \frac{1}{zx\sqrt{y}} + xy \cdot \frac{1}{xy\sqrt{z}} \right) \Rightarrow \\ & \frac{x^2}{yz\sqrt{x}} + \frac{y^2}{zx\sqrt{y}} + \frac{z^2}{xy\sqrt{z}} \geq zy \cdot \frac{1}{yz\sqrt{x}} + xz \cdot \frac{1}{zx\sqrt{y}} + xy \cdot \frac{1}{xy\sqrt{z}} = \\ & = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} \end{aligned}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

$$\begin{aligned} & \frac{1+\sqrt{a}}{bc} + \frac{1+\sqrt{b}}{ca} + \frac{1+\sqrt{c}}{ab} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} + \frac{\sqrt{a}}{bc} + \frac{\sqrt{b}}{ca} + \frac{\sqrt{c}}{ab} \geq \\ & \geq \frac{1}{ab} + \frac{1}{c\sqrt{b}} + \frac{1}{bc} + \frac{1}{a\sqrt{c}} + \frac{1}{ca} + \frac{1}{b\sqrt{a}} \geq \\ & \geq \frac{1+\sqrt{a}}{bc} + \frac{1+\sqrt{b}}{ca} + \frac{1+\sqrt{c}}{ab} \text{ true,} \end{aligned}$$



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*Because:*

$$\frac{\sqrt{a}}{bc} + \frac{\sqrt{b}}{ca} + \frac{\sqrt{c}}{ab} \geq \frac{1}{c\sqrt{b}} + \frac{1}{a\sqrt{c}} + \frac{1}{b\sqrt{a}}$$

*Iff  $a\sqrt{a} + b\sqrt{b} + c\sqrt{c} \geq b\sqrt{c} + a\sqrt{b} + c\sqrt{a}$  true if we give  $a = x^2, b = y^2, c = z^2$ .*

**Solution 3 by Tran Hong-Dong Thap-Vietnam**

*For  $x, y > 0$  we have:*

$$(x+y)\left(\frac{1}{x} + \frac{1}{y}\right) \geq 4 \Leftrightarrow \frac{4}{x+y} \leq \frac{1}{x} + \frac{1}{y}; (*)$$

*Using (\*) inequality:*

$$\frac{4}{ac+b\sqrt{a}} \leq \frac{1}{ac} + \frac{1}{b\sqrt{a}} = \frac{1}{ac} + \frac{\sqrt{b}}{ba}; (1)$$

$$\frac{4}{ab+c\sqrt{b}} \leq \frac{1}{ab} + \frac{1}{c\sqrt{b}} = \frac{1}{ab} + \frac{\sqrt{b}}{bc}; (2)$$

$$\frac{4}{bc+a\sqrt{c}} \leq \frac{1}{bc} + \frac{1}{a\sqrt{c}} = \frac{1}{bc} + \frac{\sqrt{c}}{ac}; (3)$$

*From (1), (2), (3) we have:*

$$LHS \leq \frac{1+\sqrt{a}}{bc} + \frac{1+\sqrt{b}}{ca} + \frac{1+\sqrt{c}}{ab} = RHS$$

**628. In  $\Delta ABC$  the following relationship holds:**

$$\frac{a}{a+\mu(b+c)} + \frac{b}{b+\mu(c+a)} + \frac{c}{c+\mu(a+b)} \geq \frac{3}{1+2\mu}, \mu \geq 1$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Anastase Florică-Romania**

$$\sum_{cyc} \frac{a}{a+\mu(b+c)} = \sum_{cyc} \frac{a^2}{a^2 + \mu(ab+ac)} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum a)^2}{\sum a^2 + 2\mu \sum ab} \geq \frac{3}{1+2\mu}$$

$$(1+2\mu) \left( \sum a \right)^2 \geq 3 \sum a^2 + 6\mu \sum ab$$

$$2(\mu-1) \sum a^2 \geq 2(\mu-1) \sum ab$$



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$$(\mu - 1)[(a - b)^2 + (b - c)^2 + (c - a)^2] \geq 0 \text{ true.}$$

*Equality holds if  $a = b = c \Leftrightarrow \Delta ABC$  is equilateral.*

**Solution 2 by George Florin Șerban-Romania**

$$\sum_{cyc} \frac{a}{a + \mu(b + c)} = \sum_{cyc} \frac{a^2}{a^2 + \mu(ab + ac)} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum a)^2}{\sum a^2 + 2\mu \sum ab} \geq \frac{3}{1 + 2\mu}$$

$$4s^2(1 + 2\mu) \geq 3 \sum a^2 + 6\mu \sum ab$$

$$4s^2 + 8s^2\mu \geq 6s^2 - 6r^2 - 24Rr + 6\mu(s^2 + r^2 + 4Rr)$$

$$8s^2\mu - 6s^2\mu - 6\mu r^2 - 24\mu Rr \geq 2s^2 - 6r^2 - 24Rr$$

$$2s^2\mu - 6\mu r^2 - 24\mu Rr \geq 2s^2 - 6r^2 - 24Rr$$

$$\mu(2s^2 - 6r^2 - 24Rr) \stackrel{?}{\geq} 2s^2 - 6r^2 - 24Rr$$

*We must show that:  $2s^2 - 6r^2 - 24Rr \geq 0$*

$$2s^2 \geq 6r^2 + 24Rr \Leftrightarrow s^2 \geq 3r^2 + 12Rr$$

$$s^2 \stackrel{\text{Gerretsen}}{\geq} 16Rr - 5r^2 \stackrel{?}{\geq} 3r^2 + 12Rr \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r \text{ (Euler)} \Rightarrow \mu \geq 1 \text{ true.}$$

**629. If  $0 < a, b, c < 1$  then:**

$$\prod_{cyc} \frac{(1 + ab)(1 + ac)}{1 + a\sqrt{bc}} \geq \left(1 + \sqrt[3]{a^2b^2c^2}\right)^3$$

*Proposed by Florică Anastase-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

*For  $x, y, z > 0$  we have:*

$$(1 + x)(1 + y) \geq (1 + \sqrt{xy})^2 \Leftrightarrow 1 + x + y + xy \geq 1 + 2\sqrt{xy} + xy \Leftrightarrow$$

$$x + y - 2\sqrt{xy} \geq 0 \Leftrightarrow (\sqrt{x} - \sqrt{y})^2 \geq 0 \text{ true.}$$

$$(1 + x)(1 + y)(1 + z) \geq (1^3 + \sqrt[3]{x^3})(1^3 + \sqrt[3]{y^3})(1^3 + \sqrt[3]{z^3}) \stackrel{\text{Holder}}{\geq}$$

$$\geq (1 \cdot 1 \cdot 1 + \sqrt[3]{xyz})^3 = (1 + \sqrt[3]{xyz})^3; (*)$$

*Now,*



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$$\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geq \frac{(1+\sqrt{ab \cdot ac})^2}{1+a\sqrt{bc}} = 1 + a\sqrt{bc}$$

$$\prod_{cyc} \frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geq \prod_{cyc} (1+a\sqrt{bc}) \stackrel{by(*)}{\geq} \left(1 + \sqrt[3]{a^2b^2c^2}\right)^3$$

*Proved.*

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $0 < a, b, c < 1$ , we have the following relations*

$$\begin{aligned} & (1+ab)(1+ac)(1+bc)(1+ba)(1+ca)(1+cb) \geq \\ & \geq (1+a\sqrt{bc})(1+b\sqrt{ca})(1+c\sqrt{ab})(1+a\sqrt{bc})(1+b\sqrt{ca})(1+c\sqrt{ab}) \\ & \geq (1+a\sqrt{bc})(1+b\sqrt{ca})(1+c\sqrt{ab}) \left(1 + \sqrt[3]{a^2b^2c^2}\right)^3 \end{aligned}$$

*Hence*

$$\prod_{cyc} \frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geq \left(1 + \sqrt[3]{a^2b^2c^2}\right)^3$$

**630. If  $x, y, z > 0, \mu \geq 0$  then:**

$$\sum_{cyc} \left( \frac{x^4}{x^3 + \mu y^3} \right)^3 \geq \frac{x^3 + y^3 + z^3}{(1 + \mu)^3}$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Florică Anastase-Romania**

$$\begin{aligned} \sum_{cyc} \left( \frac{x^4}{x^3 + \mu y^3} \right)^3 &= \sum_{cyc} \frac{(x^3)^4}{(x^3 + \mu y^3)^3} \stackrel{\text{Radon}}{\geq} \frac{(x^3 + y^3 + z^3)^4}{(1 + \mu)^3 (x^3 + y^3 + z^3)^3} = \\ &= \frac{x^3 + y^3 + z^3}{(1 + \mu)^3} \end{aligned}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

*If  $x, y, z: 0, \mu \geq 0$  then:*



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$$\begin{aligned}
 & \left( \frac{x^4}{x^3 + \mu y^3} \right)^3 + \left( \frac{y^4}{y^3 + \mu z^3} \right)^3 + \left( \frac{z^4}{z^3 + \mu x^3} \right)^3 = \\
 &= \frac{x^{12}}{(x^3 + \mu y^3)^3} + \frac{y^{12}}{(y^3 + \mu z^3)^3} + \frac{z^{12}}{(z^3 + \mu x^3)^3} \geq \\
 &\geq \frac{\left( \frac{x^6}{x^3 + \mu y^3} + \frac{y^6}{y^3 + \mu z^3} + \frac{z^6}{z^3 + \mu x^3} \right)^2}{x^3 + y^3 + z^3 + \mu(x^3 + y^3 + z^3)} \geq \\
 &\geq \frac{\left( \frac{(x^3 + y^3 + z^3)^2}{x^3 + y^3 + z^3 + \mu(x^3 + y^3 + z^3)} \right)^2}{x^3 + y^3 + z^3 + \mu(x^3 + y^3 + z^3)} \geq \frac{x^3 + y^3 + z^3}{(1 + \mu)^3}
 \end{aligned}$$

*Where:  $(x^3 + y^3 + z^3)^3(1 + \mu)^3 \geq (x^3 + y^3 + z^3 + \mu(x^3 + y^3 + z^3))^3$*

*$(x^3 + y^3 + z^3)(1 + \mu) \geq x^3 + y^3 + z^3 + \mu(x^3 + y^3 + z^3)$  true.*

**631. If  $a, b, c > 0$ ,  $abc = 1$  then:**

$$\frac{a}{a^5 + 1} + \frac{b}{b^5 + 1} + \frac{c}{c^5 + 1} \leq \frac{3}{2}$$

*Proposed by Jalil Hajimir-Toronto-Canada*

**Solution by Tran Hong-Dong Thap-Vietnam**

*First, for all  $x > 0$  we need to prove:*

$$\begin{aligned}
 \frac{x}{x^5 + 1} &\leq \frac{1}{x^3 + 1} \leftrightarrow x^5 + 1 \geq x(x^3 + 1) \leftrightarrow x^5 - x^4 - x + 1 \geq 0 \\
 &\leftrightarrow (x - 1)^2(x^2 + 1) \geq 0 \quad (\therefore \text{true for } x > 0)
 \end{aligned}$$

*Equality  $\leftrightarrow x = 1$ . Now,*

$$\begin{aligned}
 LHS &= \sum \frac{a}{a^5 + 1} \leq \sum \frac{1}{a^3 + 1} \stackrel{(*)}{\leq} \frac{3}{2} \\
 (*) &\leftrightarrow \frac{1}{a^3 + 1} + \frac{1}{b^3 + 1} + \frac{1}{c^3 + 1} \leq \frac{3}{2} \\
 &\leftrightarrow 2 \sum (a^3 + 1)(b^3 + 1) \leq 3(a^3 + 1)(b^3 + 1)(c^3 + 1) \\
 &\leftrightarrow 3a^3b^3c^3 + a^3b^3 + b^3c^3 + c^3a^3 - a^3 - b^3 - c^3 - 3 \geq 0
 \end{aligned}$$



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$$\begin{aligned}
 & \stackrel{abc=1}{\Leftrightarrow} 3 + a^3b^3 + b^3c^3 + c^3a^3 - a^3 - b^3 - c^3 - 3 \geq 0 \\
 & \Leftrightarrow a^3b^3 + b^3c^3 + c^3a^3 - a^3 - b^3 - c^3 \geq 0 \\
 & \stackrel{u=a^3, v=b^3, w=c^3}{\Leftrightarrow} \stackrel{(**)}{uv + vw + wu \geq u + v + w} \\
 & \stackrel{uvw=1}{\Leftrightarrow} uv + vw + wu \geq \sqrt[3]{uvw}(u + v + w) \\
 \Leftrightarrow (uv + vw + wu)^3 & \geq uvw(u + v + w)^3 \Leftrightarrow -(uw - v^2)(u^2 - vw)(uv - w^2) \geq 0 \\
 \Leftrightarrow (v^2 - uw)(u^2 - vw)(uv - w^2) & \geq 0 \stackrel{uvw=1}{\Leftrightarrow} (v^3 - 1)(u^3 - 1)(1 - w^3) \geq 0
 \end{aligned}$$

*Which is clearly true because:*

$$uvw = 1, u, v, w > 0 \rightarrow (uvw)^3 = u^3v^3w^3 = 1$$

*Dirichlet*  
 $\Rightarrow (u^3 - 1)(v^3 - 1)(w^3 - 1) \leq 0. \rightarrow (**) \text{ is true . Proved.}$

**632. If  $x, y, z > 0, xy + yz + zx = 3, n \in \mathbb{N}, n \geq 2$  then:**

$$\sqrt[n]{\frac{2x^{n+1}}{(y+z)^{3n+1}}} + \sqrt[n]{\frac{2y^{n+1}}{(z+x)^{3n+1}}} + \sqrt[n]{\frac{2z^{n+1}}{(x+y)^{3n+1}}} \geq \frac{3}{8}$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by George Florin Șerban-Romania**

$$3 = xy + yz + zx \stackrel{Am-Gm}{\geq} 3\sqrt[3]{xy \cdot yz \cdot zx} \Rightarrow 3 \geq 3\sqrt[3]{(xyz)^2} \Rightarrow xyz \leq 1$$

*Let:  $f: (0, \infty) \rightarrow \mathbb{R}, f(x) = x^{-\frac{1}{n}}, f'(x) = -\frac{1}{n}x^{-\frac{1}{n}-1}$ ,*

$$f''(x) = \frac{1}{n}\left(\frac{1}{n} + 1\right)x^{-\frac{1}{n}-2} > 0 \Rightarrow f \text{ is convex.}$$

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{f(a) + f(b) + f(c)}{3} \Rightarrow \sum_{cyc} f(a) \geq 3 \cdot f\left(\frac{\sum a}{3}\right)$$

$$\sum_{cyc} \sqrt[n]{\frac{2x^{n+1}}{(y+z)^{3n+1}}} = \sum_{cyc} \left(\frac{2x^{n+1}}{(y+z)^{3n+1}}\right)^{\frac{1}{n}} = \sum_{cyc} \left(\frac{(y+z)^{3n+1}}{2x^{n+1}}\right)^{-\frac{1}{n}}$$



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$$\begin{aligned}
 & \geq 3 \cdot \left[ \frac{1}{3} \sum_{cyc} \left( \frac{(y+z)^{3n+1}}{2x^{n+1}} \right) \right]^{-\frac{1}{n}} \stackrel{AM-GM}{\geq} 3 \cdot \left( \sqrt[3]{\prod} \frac{(y+z)^{3n+1}}{2x^{n+1}} \right)^{-\frac{1}{n}} \\
 & \stackrel{AM-GM}{\geq} 3 \cdot \left( \sqrt[3]{\prod} \frac{(2\sqrt{yz})^{3n+1}}{2x^{n+1}} \right)^{-\frac{1}{n}} = 3 \cdot \left( \sqrt[3]{\frac{8^{3n+1} \cdot (xyz)^{3n+1}}{8 \cdot (xyz)^{n+1}}} \right)^{-\frac{1}{n}} \\
 & = 3 \cdot \left( \sqrt[3]{8^{3n} \cdot (xyz)^{2n}} \right)^{-\frac{1}{n}} = \frac{3}{\sqrt[3n]{8^{3n} \cdot (xyz)^2}} \geq \frac{3}{8\sqrt[3]{1^2}} = \frac{3}{3} = \frac{8}{3}
 \end{aligned}$$

*Solution 2 by Tran Hong-Dong Thap-Vietnam*

$$\begin{aligned}
 & \bullet \quad \sqrt[n]{\frac{2x^{n+1}}{(y+z)^{3n+1}}} = \frac{x}{(y+z)^3} \cdot \sqrt[n]{\frac{2x}{y+z}} = \\
 & \frac{x}{(y+z)^3} \cdot \sqrt[n]{\underbrace{1 \cdot 1 \dots 1}_{(n-1)-number} \cdot \frac{2x}{y+z}} \stackrel{AM-HM}{\leq} \frac{x}{(y+z)^3} \cdot \frac{n}{1+1+\dots+1+\frac{1}{(\frac{2x}{y+z})}} \\
 & \bullet \quad = \frac{x}{(y+z)^3} \frac{n}{1+1+\dots+1+\frac{y+z}{2x}} = \frac{2nx^2}{(y+z)^3(2(n-1)x+y+z)} = 2n \cdot \frac{\left(\frac{x}{y+z}\right)^3}{2(n-1)x^2+xy+xz};
 \end{aligned}$$

*• Similary:*

$$\sqrt[n]{\frac{2y^{n+1}}{(z+x)^{3n+1}}} \geq 2n \cdot \frac{\left(\frac{y}{z+x}\right)^3}{2(n-1)y^2+yz+xy};$$

$$\sqrt[n]{\frac{2z^{n+1}}{(x+y)^{3n+1}}} \geq 2n \cdot \frac{\left(\frac{z}{x+y}\right)^3}{2(n-1)z^2+zx+yz};$$

*So,*

$$LHS = \sum_{cyc} \sqrt[n]{\frac{2x^{n+1}}{(y+z)^{3n+1}}} \geq$$

$$2n \cdot \sum_{cyc} \frac{\left(\frac{x}{y+z}\right)^3}{2(n-1)x^2+xy+xz} \stackrel{Holder}{\geq} 2n \cdot \frac{\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right)^3}{3 \cdot 2 \cdot ((n-1)(x^2+y^2+z^2) + (xy+yz+zx))}$$



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$$\begin{aligned}
 &= \frac{n}{3} \cdot \frac{\left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)^3}{(n-1)((x+y+z)^2 - 2(xy+yz+zx)) + (xy+yz+zx)} \\
 &= \frac{n}{3} \cdot \frac{\left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)^3}{(n-1)((x+y+z)^2 - 6) + 3} = \frac{n}{3} \cdot \frac{\left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)^3}{(n-1)(x+y+z)^2 - 6n + 9} = \Omega; \\
 \diamond \quad &\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = \frac{x^2}{yx+xz} + \frac{y^2}{yz+xy} + \frac{z^2}{xz+yz} \stackrel{\text{Schwarz}}{\geq} \frac{(x+y+z)^2}{2(xy+yz+zx)} = \frac{(x+y+z)^2}{6};
 \end{aligned}$$

◊

*Let  $t = (x+y+z)^2 \geq 3(xy+yz+zx) = 3 \cdot 3 = 9$ ;*

*We must show that:*

$$\frac{n}{3} \cdot \frac{t^3}{6^3((n-1)t - 6n + 9)} \geq \frac{3}{8};$$

$$\Leftrightarrow 8nt^3 \geq 9 \cdot 6^3 \cdot ((n-1)t - 6n + 9); \Leftrightarrow (t-9)(n(t-9)(t+18) + 243) \geq 0;$$

*Which is clearly true because:*

$$t \geq 9, n \geq 2 \quad (n \in \mathbb{N}) \rightarrow t-9 \geq 0, n(t-9)(t+18) + 243 \geq 243 > 0$$

*Proved. Equality if and only if  $x = y = z = 1$ .*

**633. If  $a, b, c > 0$  then:**

$$\sum_{cyc} \frac{c + \sqrt{ab}}{\sqrt{ab}(a+b+2c)} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned}
 &\sum_{cyc} \frac{c + \sqrt{ab}}{\sqrt{ab}(a+b+2c)} = \\
 &= \frac{c + \sqrt{ab}}{\sqrt{ab}(a+b+2c)} + \frac{b + \sqrt{ac}}{\sqrt{ac}(a+c+2b)} + \frac{a + \sqrt{bc}}{\sqrt{bc}(b+c+2a)} \\
 &= \frac{1}{\sqrt{ab}(a+b+2c)} + \frac{1}{\sqrt{ac}(a+c+2b)} + \frac{1}{\sqrt{bc}(b+c+2a)} +
 \end{aligned}$$



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$$\begin{aligned}
 & + \frac{c}{\sqrt{ab}(a+b+2c)} + \frac{b}{\sqrt{ac}(a+c+2b)} + \frac{a}{\sqrt{bc}(b+c+2a)} = \Omega \\
 & \Rightarrow \Omega \stackrel{Am-Gm}{\geq} \left( \frac{1}{a+b+2c} + \frac{1}{b+c+2a} + \frac{1}{a+c+2b} \right) \\
 & + \left( \frac{\frac{2c}{a+b}}{a+b+2c} + \frac{\frac{2b}{a+c}}{a+c+2b} + \frac{\frac{2a}{b+c}}{b+c+2a} \right) \\
 & = \frac{1 + \frac{2c}{a+b}}{a+b+2c} + \frac{1 + \frac{2b}{a+c}}{a+c+2b} + \frac{1 + \frac{2a}{b+c}}{b+c+2a} \\
 & = \frac{a+b+2c}{a+b+2c} + \frac{a+c+2b}{a+c+2b} + \frac{b+c+2a}{b+c+2a} \\
 & = \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}
 \end{aligned}$$

*Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand*

*For  $x, y, z > 0$  we have:*

$$\begin{aligned}
 \frac{z^2 + xy}{xy(x^2 + y^2 + 2z^2)} & \geq \frac{1}{x^2 + y^2} \Leftrightarrow (x^2 + y^2)(z^2 + xy) \geq xy(x^2 + y^2 + 2z^2) \\
 & \Leftrightarrow x^2z^2 + x^3y + y^2z^2 + xy^3 \geq x^3y + xy^3 + 2xyz^2 \\
 & \Leftrightarrow x^2z^2 + y^2z^2 \geq 2xyz^2
 \end{aligned}$$

*Hence:*

$$\frac{x^2 + yz}{yz(2x^2 + y^2 + z^2)} \geq \frac{1}{y^2 + z^2}; \quad \frac{y^2 + zx}{zx(x^2 + 2y^2 + z^2)} \geq \frac{1}{z^2 + x^2}$$

*Hence:*

$$\begin{aligned}
 \frac{z^2 + xy}{xy(x^2 + y^2 + 2z^2)} + \frac{x^2 + yz}{yz(2x^2 + y^2 + z^2)} + \frac{y^2 + zx}{zx(x^2 + 2y^2 + z^2)} \\
 \geq \frac{1}{x^2 + y^2} + \frac{1}{y^2 + z^2} + \frac{1}{z^2 + x^2}
 \end{aligned}$$

*For all  $a, b, c > 0$ , we give:  $a = x^2, b = y^2, c = z^2$*

$$\sum_{cyc} \frac{c + \sqrt{ab}}{\sqrt{ab}(a+b+2c)} \geq \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}$$



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**634. Let  $x, y, z \in (0, \infty)$ , prove:**

$$\frac{[x] + 1}{\sqrt{y^2 + [z^2]}} + \frac{[y] + 1}{\sqrt{z^2 + [x^2]}} + \frac{[z] + 1}{\sqrt{x^2 + [y^2]}} > 2$$

*Proposed by Jalil Hajimir-Toronto-Canada*

**Solution by George Florin Șerban-Romania**

$$t - 1 < [t] \leq t, \forall t \in \mathbb{R}$$

$z^2 - 1 < [z^2] \leq z^2 \text{ and analogs.}$

$$\begin{aligned} \sum_{cyc} \frac{[x] + 1}{\sqrt{y^2 + [z^2]}} &> \sum_{cyc} \frac{[x] + 1}{\sqrt{y^2 + z^2}} > \sum_{cyc} \frac{x}{\sqrt{y^2 + z^2}} = \sum_{cyc} \sqrt{\frac{x^2}{y^2 + z^2}} = \sum_{cyc} \sqrt{\frac{x^2}{y^2 + z^2}} \cdot 1 \\ &\stackrel{Am-Hm}{\geq} \sum_{cyc} \frac{\frac{2x^2}{y^2 + z^2}}{\frac{x^2}{y^2 + z^2} + 1} = \sum_{cyc} \frac{2x^2}{x^2 + y^2 + z^2} = 2 \end{aligned}$$

**635. If  $x, y, z > 0, xyz = 1, n \in \mathbb{N} - \{0\}$  then:**

$$x^n \sqrt{\frac{2}{y+z}} + y^n \sqrt{\frac{2}{z+x}} + z^n \sqrt{\frac{2}{x+y}} \geq 3$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Marian Ursărescu-Romania**

$$\sqrt{\frac{2}{y+z}} = \sqrt{\frac{2}{y+z}} \cdot 1 \geq \frac{2}{\frac{y+z}{2} + 1} = \frac{4}{y+z+2}$$

$$\text{We must show: } \frac{x^n}{y+z+2} + \frac{y^n}{z+x+2} + \frac{z^n}{x+y+2} \geq \frac{3}{4}; \quad (1)$$

*From Holder inequality we have:*

$$\frac{x^n}{y+z+2} + \frac{y^n}{z+x+2} + \frac{z^n}{x+y+2} \geq \frac{(x+y+z)^n}{3^{n-2} \cdot 2(x+y+z+3)}; \quad (2)$$

*From (1),(2) we must show:*

$$\frac{(x+y+z)^n}{3^{n-2} \cdot 2(x+y+z+3)} \geq \frac{3}{4} \Leftrightarrow 2(x+y+z)^n \geq 3^{n-1}(x+y+z+3); \quad (3)$$



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*Because  $xyz = 1 \Rightarrow \exists a, b, c > 0$  such that  $x = \frac{a^2}{bc}, y = \frac{b^2}{ca}, z = \frac{c^2}{ab} \Leftrightarrow$*

$$2\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right)^n \geq 3^{n-1} \left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} + 3\right) \Leftrightarrow$$

$$2\left(\frac{a^3 + b^3 + c^3}{abc}\right)^n \geq 3^{n-1} \left(\frac{a^3 + b^3 + c^3 + 3abc}{abc}\right) \Leftrightarrow$$

$$2(a^3 + b^3 + c^3)^n \geq 3^{n-1}(abc)^{n-1}(a^3 + b^3 + c^3) + 3^n(abc)^n; \quad (4)$$

$$\begin{aligned} \text{But } (a^3 + b^3 + c^3)^n &\geq (3abc)^n; \quad (5) \text{ and } (a^3 + b^3 + c^3)^{n-1} \geq 3^{n-1}(abc)^{n-1} \\ &\Rightarrow (a^3 + b^3 + c^3)^n \geq 3^{n-1}(abc)^{n-1}(a^3 + b^3 + c^3); \quad (6) \end{aligned}$$

*From (5),(6) we get (4) true.*

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $x, y, z > 0, xyz = 1$ , we will receive the nexts*

$$x\sqrt{y+z} + y\sqrt{z+x} + z\sqrt{x+y} \leq \frac{(x+y+z)(\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x})}{3}$$

$$\begin{aligned} \text{Consider } x^n \sqrt{\frac{2}{y+z}} + y^n \sqrt{\frac{2}{z+x}} + z^n \sqrt{\frac{2}{x+y}} &= \sqrt{2} \left( x^{n-1} \cdot \frac{x}{\sqrt{y+z}} + y^{n-1} \cdot \frac{y}{\sqrt{z+x}} + z^{n-1} \cdot \frac{z}{\sqrt{x+y}} \right) \\ &\geq \sqrt{2} \frac{(x^{n-1} + y^{n-1} + z^{n-1}) \left( \frac{x}{\sqrt{y+z}} + \frac{y}{\sqrt{z+x}} + \frac{z}{\sqrt{x+y}} \right)}{3} \stackrel{x^{n-1}+y^{n-1}+z^{n-1} \geq 3}{\geq} \\ &\geq \sqrt{2} \left( \frac{x}{\sqrt{y+z}} + \frac{y}{\sqrt{z+x}} + \frac{z}{\sqrt{x+y}} \right) \geq \frac{\sqrt{2}(x+y+z)^2}{x\sqrt{y+z} + y\sqrt{z+x} + z\sqrt{x+y}} \\ &\geq \frac{\sqrt{2}(x+y+z)^2}{(\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x}) \frac{(x+y+z)}{3}} = \frac{3(x+y+z)}{\sqrt{\frac{x+y}{2}} + \sqrt{\frac{y+z}{2}} + \sqrt{\frac{z+x}{2}}} \\ &= \frac{3(x+y+z)}{\sqrt{3(x+y+z)}} \geq \frac{3\sqrt{3(x+y+z)}}{\sqrt{3(x+y+z)}} = 3 \end{aligned}$$

**636. If  $a, b, c > 0$  then:**

$$\frac{1}{a+ab+b} + \frac{1}{b+bc+c} + \frac{1}{c+ca+a} \leq \sqrt{\frac{a^2 + b^2 + c^2}{3a^2b^2c^2}}$$

*Proposed by Daniel Sitaru-Romania*



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**Solution 1 by Florică Anastase-Romania**

$$\begin{aligned}
 a + ab + b &\geq 3\sqrt[3]{a^2b^2} \rightarrow \frac{1}{a + ab + b} \leq \frac{1}{3\sqrt[3]{a^2b^2}} \rightarrow \\
 \frac{1}{a + ab + b} + \frac{1}{b + bc + c} + \frac{1}{c + ca + a} &\leq \frac{1}{3} \left( \frac{1}{\sqrt[3]{a^2b^2}} + \frac{1}{\sqrt[3]{b^2c^2}} + \frac{1}{\sqrt[3]{c^2a^2}} \right) \\
 &\leq \sqrt{\frac{1}{3} \left( \frac{1}{a^2b^2} + \frac{1}{b^2c^2} + \frac{1}{c^2a^2} \right)} = \sqrt{\frac{a^2 + b^2 + c^2}{3a^2b^2c^2}}
 \end{aligned}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $a, b, c > 0$  we have:

$$(ac + bc + c + ac + ab + a + ab + bc + b)^2 \leq 27(a^2 + b^2 + c^2)$$

$$\begin{aligned}
 ac + bc + c + ac + ab + a + ab + bc + b &\leq 9 \cdot \sqrt{\frac{a^2 + b^2 + c^2}{3}} \\
 \frac{1}{9} \left( \frac{c}{\frac{1}{a}} + \frac{c}{\frac{1}{b}} + \frac{c}{\frac{1}{c}} + \frac{a}{\frac{1}{c}} + \frac{a}{\frac{1}{b}} + \frac{a}{\frac{1}{a}} + \frac{b}{\frac{1}{a}} + \frac{b}{\frac{1}{c}} + \frac{b}{\frac{1}{b}} \right) &\leq \sqrt{\frac{a^2 + b^2 + c^2}{3}}
 \end{aligned}$$

Hence

$$\begin{aligned}
 \frac{c}{\frac{1}{a} + \frac{1}{b} + 1} + \frac{a}{\frac{1}{b} + \frac{1}{c} + 1} + \frac{b}{\frac{1}{a} + \frac{1}{c} + 1} &\leq \sqrt{\frac{a^2 + b^2 + c^2}{3}} \\
 \frac{abc}{a + ab + b} + \frac{abc}{b + bc + c} + \frac{abc}{a + ac + c} &\leq \sqrt{\frac{a^2 + b^2 + c^2}{3}} \\
 \frac{1}{a + ab + b} + \frac{1}{b + bc + c} + \frac{1}{c + ca + a} &\leq \sqrt{\frac{a^2 + b^2 + c^2}{3a^2b^2c^2}}
 \end{aligned}$$

**637. If  $a, b, c > 0, abc = 4^n, n \in \mathbb{N}^*$  then:**

$$\sum_{cyc} \frac{(n+1)(b^{2n+3} + c^{2n+3}) + a^{2n+3}}{(b+c)^{2n}} \geq 3(2n+3)$$

*Proposed by Marin Chirciu-Romania*



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**Solution by Tran Hong-Dong Thap-Vietnam**

$$\text{For } n \in \mathbb{N}^*, a, b, c > 0: b^{2n+3} + c^{2n+3} \geq \frac{(b+c)^{2n+3}}{2^{2n+2}} \Rightarrow$$

$$\frac{(n+1)(b^{2n+3} + c^{2n+3})}{(b+c)^{2n}} \geq \frac{n+1}{4^{n+1}} \cdot (b+c)^3$$

$$\sum_{cyc} \frac{(n+1)(b^{2n+3} + c^{2n+3})}{(b+c)^{2n}} \geq \frac{n+1}{4^{n+1}} \cdot [(a+b)^3 + (b+c)^3 + (c+a)^3] \geq$$

$$\geq \frac{n+1}{4^{n+1}} \cdot \frac{8}{9} \cdot (a+b+c)^3 \geq \frac{n+1}{4^{n+1}} \cdot \frac{8}{9} \cdot 27abc = \frac{n+1}{4^{n+1}} \cdot \frac{8}{9} \cdot 4^n$$

$$= 6(n+1); (*)$$

$$\text{Suppose: } a \geq b \geq c \Rightarrow a^3 \geq b^3 \geq c^3; \frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b} \Rightarrow$$

$$\left(\frac{a}{b+c}\right)^{2n} \geq \left(\frac{b}{c+a}\right)^{2n} \geq \left(\frac{c}{a+b}\right)^{2n}$$

$$\Rightarrow \frac{a^{2n+3}}{(b+c)^{2n}} + \frac{b^{2n+3}}{(c+a)^{2n}} + \frac{c^{2n+3}}{(a+b)^{2n}}$$

$$\stackrel{\text{Cebyshev}}{\geq} \frac{1}{3}(a^3 + b^3 + c^3) \left( \left(\frac{a}{b+c}\right)^{2n} + \left(\frac{b}{c+a}\right)^{2n} + \left(\frac{c}{a+b}\right)^{2n} \right)$$

$$\geq \frac{1}{3} \cdot (3abc) \cdot \left( \frac{\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^{2n}}{3^{2n-1}} \right) \stackrel{\text{Nesbitt}}{\geq} 4^n \cdot \frac{\left(\frac{3}{2}\right)^{2n}}{3^{2n-1}} = 3; (**)$$

$$(*) + (**) \Rightarrow \sum_{cyc} \frac{(n+1)(b^{2n+3} + c^{2n+3}) + a^{2n+3}}{(b+c)^{2n}} = 6(n+1) + 3 \geq 3(2n+3)$$

**638. If  $x, y, z > 0, xy + yz + zx = 3$  then:**

$$\sqrt{\frac{2x}{(y+z)^5}} + \sqrt{\frac{2y}{(z+x)^5}} + \sqrt{\frac{2z}{(x+y)^5}} \geq \frac{3}{4}$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

With  $x, y, z > 0$ , let:  $t = xy + yz + zx \geq \sqrt{3(xy + yz + zx)} = 3$



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$$\begin{aligned}
 \sqrt{\frac{2x}{(y+z)^5}} &= \frac{1}{(y+z)^2} \cdot \sqrt{\frac{2x}{y+z}} = \frac{1}{(y+z)^2} \cdot \sqrt{1 \cdot \frac{2x}{y+z}} \stackrel{Am-Hm}{\geq} \\
 &\geq \frac{1}{(y+z)^2} \cdot \frac{2}{1 + \frac{1}{\frac{2x}{y+z}}} = \frac{1}{(y+z)^2} \cdot \frac{2}{1 + \frac{y+z}{2x}} = \\
 &= \frac{4x}{(y+z)^2(2x+y+z)} = \frac{4 \left( \frac{x}{y+z} \right)^2}{2x^2 + xy + xz}
 \end{aligned}$$

*Similary:*  $\sqrt{\frac{2y}{(z+x)^5}} \geq \frac{4 \left( \frac{y}{z+x} \right)^2}{2y^2 + yz + yz}$  and  $\sqrt{\frac{2z}{(x+y)^5}} \geq \frac{4 \left( \frac{z}{x+y} \right)^2}{2z^2 + zx + zy}$

$$\begin{aligned}
 LHS &= \sum_{cyc} \sqrt{\frac{2x}{(y+z)^5}} \geq \\
 &\geq 4 \left[ \frac{\left( \frac{x}{y+z} \right)^2}{2x^2 + xy + xz} + \frac{\left( \frac{y}{z+x} \right)^2}{2y^2 + yz + yz} + \frac{\left( \frac{z}{x+y} \right)^2}{2z^2 + zx + zy} \right] \\
 &\stackrel{Bergstrom}{\geq} 4 \cdot \frac{\left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)^2}{2(x^2 + y^2 + z^2 + xy + yz + zx)} \\
 &= \frac{2 \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)^2}{(x^2 + y^2 + z^2 + xy + yz + zx)} = \frac{2 \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)^2}{(x+y+z)^2 - 3}
 \end{aligned}$$

$$\begin{aligned}
 \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} &= \frac{x^2}{xy+xz} + \frac{y^2}{yz+yx} + \frac{z^2}{zx+zy} \stackrel{Bergstrom}{\geq} \\
 &\geq \frac{(x+y+z)^2}{2(xy+yz+zx)} = \frac{(x+y+z)^2}{6}
 \end{aligned}$$

*So, we need to prove:*  $\frac{2 \frac{t^4}{6^2}}{t^2-3} \geq \frac{3}{4} \Leftrightarrow 2t^4 \geq 27(t^2-3) \Leftrightarrow 2t^4 - 27t^2 + 81 \geq 0$

$\Leftrightarrow (t^2-9)(2t^2-9) \geq 0$  true for  $t \geq 3 \Leftrightarrow t^2-9 \geq 0; 2t^2-9 \geq 9 > 0$

*Proved. Equality*  $\Leftrightarrow x = y = z = 1$ .



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**639. If  $a, b, c > 0, \mu \leq 1$  then:**

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \mu \cdot \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}} \geq \frac{\mu+3}{2}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \mu \cdot \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}} &\geq \frac{\mu+3}{2} \\ \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\stackrel{(*)}{\geq} \frac{\mu+3}{2} - \mu \cdot \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}} \end{aligned}$$

*More, by Schur's inequality:*

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{4abc}{(a+b)(b+c)(c+a)} &\geq 2 \\ \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\stackrel{(**)}{\geq} 2 - \frac{4abc}{(a+b)(b+c)(c+a)} \end{aligned}$$

*From (\*), (\*\*) we need to prove:*

$$\begin{aligned} 2 - \frac{4abc}{(a+b)(b+c)(c+a)} &\geq \frac{\mu+3}{2} - \mu \cdot \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}} \\ \frac{1-\mu}{2} &\stackrel{(1)}{\geq} 2 \cdot \frac{2abc}{(a+b)(b+c)(c+a)} - \mu \cdot \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}} \end{aligned}$$

*For  $a, b, c > 0 \Rightarrow (a+b)(b+c)(c+a) \stackrel{Am-Gm}{\geq} 8abc$  (Cesaro)*

$$\Rightarrow 0 < \frac{2abc}{(a+b)(b+c)(c+a)} \leq \frac{1}{4}$$

$$\text{Let: } t = \frac{2abc}{(a+b)(b+c)(c+a)} \Rightarrow 0 < t \leq \frac{1}{4}$$

$$(1) \Leftrightarrow 2t^2 - \mu t - \left(\frac{1-\mu}{2}\right) \leq 0 \Leftrightarrow \frac{1}{2}(1-2t)(\mu - (1+2t)) \leq 0$$

*Which is clearly true because:*



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$$0 < t \leq \frac{1}{2}; \mu \leq 1 \Rightarrow 1 + 2t > 1 \geq \mu \Rightarrow \mu - (1 + 2t) < 0. \text{ Proved.}$$

*Equality holds if  $a = b = c$ .*

**640. If  $a, b, c, d > 0, a + b + c + d = 1$  then:**

$$\frac{a^2 + b + ad}{b + c} + \frac{b^2 + c + ba}{c + d} + \frac{c^2 + d + cb}{d + a} + \frac{d^2 + a + dc}{a + b} \geq 3$$

*Proposed by Marin Chirciu – Romania*

**Solution by Marian Ursărescu – Romania**

$$\begin{aligned} \frac{a^2 + b + ad}{b + c} &= \frac{a(a + d) + b}{b + c} = \frac{a(1 - b - c) + b}{b + c} = \frac{a + b - (b + c)a}{b + c} = \\ &= \frac{a+b}{b+c} - a \text{ and similarly } \Rightarrow \text{we must show:} \\ &\quad \frac{a+b}{b+c} + \frac{b+c}{c+d} + \frac{c+d}{d+a} + \frac{d+a}{a+b} \geq 4 \quad (1) \\ \frac{a+b}{b+c} + \frac{b+c}{c+d} + \frac{c+d}{d+a} + \frac{d+a}{a+b} &= \frac{(a+b)^2}{(a+b)(b+c)} + \frac{(b+c)^2}{(c+d)(b+c)} + \frac{(c+d)^2}{(d+a)(c+d)} + \\ &+ \frac{(d+a)^2}{(a+b)(d+a)} \stackrel{\text{Bergstrom}}{\geq} \frac{4(a+b+c+d)^2}{(a+b)(b+c) + (c+d)(b+c) + (d+a)(c+d) + (a+b)(d+a)} = \\ &= \frac{4(a+b+c+d)^2}{(a+b+c+d)^2} = 4 \Rightarrow (1) \text{ it is true.} \end{aligned}$$

**641. If  $a, b, c, d > 1, abcd = e^4$  then:**

$$\frac{\log\left(\frac{e^2}{a}\right) \cdot \log\left(\frac{e^2}{b}\right) \cdot \log\left(\frac{e^2}{c}\right) \cdot \log\left(\frac{e^2}{d}\right)}{\log(ab) \cdot \log(bc) \cdot \log(cd) \cdot \log(da)} \leq \frac{1}{16}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Rahim Shahbazov-Baku-Azerbaijan**

$$abcd = e^4 \rightarrow \log a + \log b + \log c + \log d = 4$$

$$a, b, c, d > 1 \rightarrow \log a, \log b, \log c, \log d > 0$$

Let:  $x = \log a, y = \log b, z = \log c, t = \log d \rightarrow$

$$\frac{(2-x)(2-y)(2-z)(2-t)}{(x+y)(y+z)(z+t)(t+x)} \leq \frac{1}{16}$$

$$\rightarrow (x+y)(y+z)(z+t)(t+x) \geq (4-2x)(4-2y)(4-2z)(4-2t) \rightarrow$$



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$$(x+y)(y+z)(z+t)(t+x) \geq$$

$$\geq (z+y+t-x)(z+x+t-y)(z+x+y-t)(x+y+t-z)$$

$$\begin{cases} y+z+t-x = A > 0 \\ z+x+t-y = B > 0 \\ z+x+y-t = C > 0 \\ x+y+t-z = D > 0 \end{cases} \rightarrow \begin{cases} 2(z+t) = A+B \\ 2(x+y) = C+D \\ 2(x+t) = B+D \\ 2(y+z) = A+C \end{cases}$$

$$\rightarrow (A+B)(C+D)(B+D)(A+C) \geq 16ABCD \text{ true from Am-Gm.}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

If  $a, b, c, d > 1$ ,  $abcd = e^4$  then:

$$\frac{\log\left(\frac{e^2}{a}\right) \cdot \log\left(\frac{e^2}{b}\right) \cdot \log\left(\frac{e^2}{c}\right) \cdot \log\left(\frac{e^2}{d}\right)}{\log(ab) \cdot \log(bc) \cdot \log(cd) \cdot \log(da)} \leq \frac{1}{16}$$

$$\begin{aligned} 16(2 - \log a)(2 - \log b)(2 - \log c)(2 - \log d) &\leq \log a \log b \log c \log d \\ (4 - \log a^2)(4 - \log b^2)(4 - \log c^2)(4 - \log d^2) &\leq \\ \leq (4 - \log(ab))(4 - \log(bc))(4 - \log(cd))(4 - \log(da)) \text{ true from} \\ (4 - \log a^2)(4 - \log b^2) &\leq (4 - \log(ab))^2 \\ (4 - \log c^2)(4 - \log d^2) &\leq (4 - \log(cd))^2 \\ (4 - \log a^2)(4 - \log d^2) &\leq (4 - \log(ad))^2 \\ (4 - \log c^2)(4 - \log b^2) &\leq (4 - \log(bc))^2 \end{aligned}$$

$$\text{Consider: } (4 - \log a^2)(4 - \log b^2) \leq (4 - \log(ab))^2$$

$$16 - 4\log a^2 - 4\log b^2 + \log a^2 \log b^2 \leq 16 + \log^2(ab) - 8\log(ab)$$

$$\log a^2 \log b^2 + 8\log a \log b \leq \log^2(ab) + 4\log a^2 + 4\log b^2$$

$$4\log a \log b + 8(\log a + \log b) \leq (\log a + \log b)^2 + 8(\log a + \log b)$$

$$4\log a \log b \leq \log^2 a + \log^2 b + 2\log a \log b$$

$$\log^2 a + \log^2 b \geq 2\log a \log b$$

true.

**642. If  $a, b, c, d \geq e$ ,  $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ , then:**

$$5\log(ae) \cdot \log(be) \cdot \log(ce) \cdot \log(de) \geq \log(abcd)^{16}$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**



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*For  $a, b, c, d \geq 1 \Rightarrow x = \log a \geq 1, y = \log b \geq 1, z = \log c \geq 1,$*

*$t = \log d \geq 1.$  We have:*

$$5\log(ae) \cdot \log(be) \cdot \log(ce) \cdot \log(de) \geq \log(abcd)^{16} \Leftrightarrow$$

$$5[(\log a + 1)(\log b + 1)(\log c + 1)(\log d + 1)] \geq$$

$$\geq 16(\log a + \log b + \log c + \log d + 1)$$

$$5(x+1)(y+1)(z+1)(t+1) \geq 16(x+y+z+t+1) \Leftrightarrow$$

$$5(xyzt + xyz + yzt + xyt + xzt + xy + yz + zt + xz + ty + xt) \stackrel{(*)}{\geq}$$

$$\geq 11(x+y+z+t)$$

*Because:  $x, y, z, t \geq 1 \Rightarrow (x-1)(y-1)(z-1)(t-1) \geq 0 \Leftrightarrow$*

$$xyzt \geq xyz + xzt + yzt + xyt - xy - yz - zt - xz - ty - xt + x + y + z + t - 1$$

$$5(xyzt + xyz + yzt + xyt + xzt + xy + yz + zt + xz + ty + xt) \geq$$

$$\geq 5[2(xyz + yzt + xyt + xzt) + x + y + z + t - 1]$$

*So, we need to prove:*

$$10(xyz + yzt + xyt + xzt) \geq 6(x+y+z+t) + 16$$

$$\begin{aligned} \text{But: } xyz + yzt + xyt + xzt &\stackrel{x,y,z,t \geq 1}{\geq} x \cdot 1 \cdot 1 + y \cdot 1 \cdot 1 + z \cdot 1 \cdot 1 + t \cdot 1 \cdot 1 = \\ &= x + y + z + t \Rightarrow \end{aligned}$$

$$10(xyz + yzt + xyt + xzt) \stackrel{(*)}{\geq} 10(x+y+z+t) \geq$$

$$\geq 6(x+y+z+t) + 16$$

$$(*) \Leftrightarrow 4(x+y+z+t) \geq 16 \Leftrightarrow x+y+z+t \geq 4$$

*Which is true, because:  $x, y, z, t \geq 1 \Rightarrow x+y+z+t \geq 4 \Rightarrow (*) \text{ is true. Proved.}$*

**643. If  $0 < x_i < 1, i \in \overline{1, n}, n \in \mathbb{N} - \{0\}$  then:**

$$\frac{1}{n} \sum_{i=1}^n \frac{x_i}{\sqrt{1-x_i}} \geq \frac{\sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}}{\sqrt{1 - \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}}}$$

*Proposed by Rajeev Rastogi-India*

**Solution 1 by Adrian Popa-Romania**

*Let be the function  $f(x) = \frac{x}{\sqrt{1-x}}, x \in (0, 1)$*



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$$f'(x) = \frac{2-x}{2(1-x)^{\frac{3}{2}}}; f''(x) = \frac{(4-x)\sqrt{1-x}}{4(1-x)^3} > 0, \forall x \in (0,1) \Rightarrow f - \text{convexe}$$

*From Jensen inequality, we have:*

$$\frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)$$

$$\frac{1}{n} \sum_{i=1}^n \frac{x_i}{\sqrt{1-x_i}} \geq \frac{\frac{x_1 + x_2 + \dots + x_n}{n}}{\sqrt{1 - \frac{x_1 + x_2 + \dots + x_n}{n}}} \stackrel{AGM}{\geq} \frac{\sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}}{\sqrt{1 - \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}}}$$

*Which is true because:*

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n} \Rightarrow$$

$$1 - \frac{x_1 + x_2 + \dots + x_n}{n} \leq 1 - \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}$$

$$\frac{1}{1 - \frac{x_1 + x_2 + \dots + x_n}{n}} \geq \frac{1}{1 - \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\frac{1}{n} \sum_{i=1}^n \frac{x_i}{\sqrt{1-x_i}} \stackrel{AGM}{\geq} \frac{1}{n} \cdot n \cdot \sqrt[n]{\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n \sqrt{1-x_i}}} = \frac{\sqrt[n]{\prod_{i=1}^n x_i}}{\sqrt[n]{\prod_{i=1}^n \sqrt{1-x_i}}} \stackrel{(*)}{\geq}$$

$$\geq \frac{\sqrt[n]{\prod_{i=1}^n x_i}}{\sqrt{1 - \sqrt[n]{\prod_{i=1}^n x_i}}}$$

$$(*) \Leftrightarrow \sqrt{1 - \sqrt[n]{\prod_{i=1}^n x_i}} \geq \sqrt[n]{\prod_{i=1}^n \sqrt{1-x_i}} \Leftrightarrow \sqrt{1 - \sqrt[n]{\prod_{i=1}^n x_i}} \geq \sqrt{\sqrt[n]{\prod_{i=1}^n \sqrt{1-x_i}}}$$

$$\Leftrightarrow 1 - \sqrt[n]{\prod_{i=1}^n x_i} \geq \sqrt[n]{\prod_{i=1}^n \sqrt{1-x_i}} \Leftrightarrow \sqrt[n]{\prod_{i=1}^n \sqrt{1-x_i}} + \sqrt[n]{\prod_{i=1}^n x_i} \leq 1 \Leftrightarrow$$

*Which is true because:*



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$$\begin{aligned}
 \sqrt[n]{\prod_{i=1}^n \sqrt{1-x_i}} &\stackrel{AGM}{\leq} \frac{\sum_{i=1}^n (1-x_i)}{n} = \frac{n - \sum_{i=1}^n x_i}{n} = 1 - \frac{\sum_{i=1}^n x_i}{n} \stackrel{AGM}{\leq} \\
 &\leq 1 - \sqrt[n]{\prod_{i=1}^n x_i} \Rightarrow \sqrt[n]{\prod_{i=1}^n \sqrt{1-x_i}} \leq 1 - \sqrt[n]{\prod_{i=1}^n x_i} \\
 &\Leftrightarrow \sqrt[n]{\prod_{i=1}^n \sqrt{1-x_i}} + \sqrt[n]{\prod_{i=1}^n x_i} \leq 1
 \end{aligned}$$

**644. If  $x, y, z > 0, 3(xy + yz + zx) = 1$  then:**

$$27 \sum_{cyc} x^3y + 36 \sum_{cyc} x^2y + 6 \sum_{cyc} x \geq 11$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Sanong Huayrerai-Nakon Pathom-Thailand**

**For  $x, y, z > 0, 3(xy + yz + zx) = 1$  we get:**

$$\begin{aligned}
 &27(x^3y + y^3z + z^3x) + 36(x^2y + y^2z + z^2x) + 6(x + y + z) \stackrel{CEBYSHEV}{\geq} \\
 &\geq \frac{27(x^2+y^2+z^2)(xy+yz+zx)}{3} + \frac{36(x+y+z)(xy+yz+zx)}{3} + 6(x + y + z) \\
 &= \frac{3(x^2 + y^2 + z^2)}{3} + \frac{12(x + y + z)}{3} + 6(x + y + z) \geq 3 \cdot \frac{1}{3} + 4 + 6 = 11
 \end{aligned}$$

**Because:**  $3(xy + yz + zx) = 1 \Rightarrow xy + yz + zx = \frac{1}{3}$

$$\Rightarrow (x + y + z)^2 \geq 1 \Rightarrow x + y + z \geq 1 \Rightarrow x^2 + y^2 + z^2 \geq \frac{1}{3}$$

**645. If  $a, b, c > 0, \mu \geq 0$  then:**

$$\frac{a^3}{a + \mu b} + \frac{b^3}{b + \mu c} + \frac{c^3}{c + \mu a} + \frac{\mu + 1}{9} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right) \geq 2$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**



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$$\begin{aligned}
 \frac{a^3}{a + \mu b} + \frac{b^3}{b + \mu c} + \frac{c^3}{c + \mu a} &\stackrel{\text{Holder}}{\geq} \frac{(a + b + c)^3}{3[(a + b + c) + \mu(a + b + c)]} = \\
 &= \frac{(a + b + c)^3}{3(1 + \mu)(a + b + c)} = \frac{(a + b + c)^2}{3(1 + \mu)} \\
 \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} &\stackrel{\text{Bergstrom}}{\geq} \frac{9}{ab + bc + ca} \stackrel{(a+b+c)^2 \geq 3(ab+bc+ca)}{\geq} \\
 &\geq 9 \cdot \frac{3}{(a + b + c)^2} = \frac{27}{(a + b + c)^2} \Rightarrow \\
 \frac{\mu + 1}{9} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right) &\geq \frac{3(1 + \mu)}{(a + b + c)^2} \Rightarrow \\
 LHS = \sum_{cyc} \frac{a^3}{a + \mu b} + \frac{\mu + 1}{9} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right) &\geq \frac{(a + b + c)^2}{3(1 + \mu)} + \frac{3(1 + \mu)}{(a + b + c)^2} \stackrel{\text{AGM}}{\geq} 2
 \end{aligned}$$

**646. Let  $a, b, c > 0$  such that:**

$$\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq 1. \text{ Prove that:}$$

$$a + b + c \geq ab + bc + ca$$

*Proposed by Hung Nguyen Viet-Hanoi-Vietnam*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\begin{aligned}
 (2a^2 + bc) \left( \frac{(b + c)^2}{2} + bc \right) &= \left[ (\sqrt{2}a)^2 + (\sqrt{bc})^2 \right] \left( \frac{(b + c)^2}{\sqrt{2}^2} + \sqrt{bc} \right) \stackrel{\text{BCS}}{\geq} \\
 &\geq \left( \sqrt{2}a \cdot \frac{b + c}{\sqrt{2}} + \sqrt{bc} \cdot \sqrt{bc} \right)^2 = (ab + bc + ca)^2 \Rightarrow \\
 \frac{1}{2a^2 + bc} &\leq \frac{\frac{(b + c)^2}{2} + bc}{(ab + bc + ca)^2}
 \end{aligned}$$

*Similary:*

$$\begin{aligned}
 \frac{1}{2b^2 + ca} &\leq \frac{\frac{(a + c)^2}{2} + ac}{(ab + bc + ca)^2} \\
 \frac{1}{2c^2 + ab} &\leq \frac{\frac{(a + b)^2}{2} + ab}{(ab + bc + ca)^2}
 \end{aligned}$$



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$$\text{We have: } \frac{1}{2a^2+bc} + \frac{1}{2b^2+ca} + \frac{1}{2c^2+ab} \geq 1 \Rightarrow$$

$$\frac{\frac{(a+b)^2}{2} + ab}{(ab+bc+ca)^2} + \frac{\frac{(b+c)^2}{2} + bc}{(ab+bc+ca)^2} + \frac{\frac{(a+c)^2}{2} + ac}{(ab+bc+ca)^2} \geq 1$$

$$\frac{(a+b)^2}{2} + ab + \frac{(b+c)^2}{2} + bc + \frac{(a+c)^2}{2} + ac \geq (ab+bc+ca)^2$$

$$\frac{a^2+b^2}{2} + 2ab + \frac{b^2+c^2}{2} + 2bc + \frac{c^2+a^2}{2} + 2ca \geq (ab+bc+ca)^2$$

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \geq (ab+bc+ca)^2$$

$$(a+b+c)^2 \geq (ab+bc+ca)^2$$

$$a+b+c \geq ab+bc+ca$$

**647. If  $a, b, c > 0, abc = 1$  then:**

$$\sum_{cyc} \frac{(a^{10} + b^{10})(a^9 + b^9)}{(a^4 + b^4)(a^3 + b^3)} \geq 3$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Sanong Huayrerai-Nakon Pathom-Thailand**

$$\begin{aligned} & \sum_{cyc} \frac{(a^{10} + b^{10})(a^9 + b^9)}{(a^4 + b^4)(a^3 + b^3)} \stackrel{\text{CEBYSHEV}}{\geq} \\ & \geq \sum_{cyc} \frac{(a^4 + b^4)(a^3 + b^3)(a^6 + b^6)(a^6 + b^6)}{4(a^4 + b^4)(a^3 + b^3)} = \sum_{cyc} \frac{(a^6 + b^6)^2}{4} \geq \\ & \geq \frac{\left(\sum \frac{a^6 + b^6}{2}\right)^2}{3} = \frac{(a^6 + b^6 + c^6)^2}{3} \geq 3 \end{aligned}$$

If  $abc = 1 \Rightarrow a + b + c \geq 3 \Rightarrow a^6 + b^6 + c^6 \geq 3$ . Proved

**648. If  $a, b, c > 0; abc = 1$  then:**

$$\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} \geq 1$$

*Proposed by Jalil Hajimir-Toronto-Canada*

**Solution by Michael Sterghiou-Greece**



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$$\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} \geq 1; (1)$$

*Let  $(p, q, r) = (\sum a, \sum ab, \prod a), r = 1$ . By expanding (1) we get:*

$$\frac{\sum(7-6a)(2+b^2)(2+c^2)}{\prod(2+a^2)} \geq 1 \text{ which after same computation reduces to:}$$

$$24p^2 - 12pq - 34p + 5q^2 - 54q + 111 \geq 0$$

*Or  $6(p-q)^2 + f(q) \geq 0$  where  $f(q) = 18p^2 - 34p - q^2 - 54q + 111$*

*Note that:*

$$\sum ab^2 + \sum a^2b = pq - 3; \quad \sum a^2b^2 = q^2 - 2p; \quad \sum a^2 = p^2 - 2q \text{ as}$$

$$r = 1$$

*$f(q)$  is a decreasing function of  $q$ . Assuming  $a \leq b \leq c$  (WLOG)*

*Wish  $p$  fixed  $f(q)$  becomes minimal when  $a = b (\leq 1)$  in which case if is enough that*

$$f(q) \geq 0.$$

*Wish  $a = b = x, c = \frac{1}{x^2}, p = 2x + \frac{1}{x^2}, q = \frac{2}{x} + x^2$  and*

$$f(q) \rightarrow f(x) = -\frac{1}{x^4}(x-1)^2(x^6 + 2x^5 - 15x^4 + 40x^3 - 16x^2 - 36x - 18) = \\ = -\frac{(x-1)^2}{x^4} \cdot \sigma(x), \text{ where}$$

$$\sigma(x) = x^4 \left( \underbrace{x^2 + 2x - 3}_{<0} \right) + x \left( \underbrace{40x^2 - 16x - 24}_{<0} \right) - 12x^4 - 12x - 18 < 0$$

*For  $0 \leq x \leq 1$ . Hence  $g(x) \geq 0$ .*

*With equality for  $x = a = b = 1, c = 1$ . Done.*

**649. If  $a, b, c > 0, \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 12$  then:**

$$\frac{(a+b+\sqrt{ab})^3}{(a+b)^2} + \frac{(b+c+\sqrt{bc})^3}{(b+c)^2} + \frac{(c+a+\sqrt{ca})^3}{(c+a)^2} \geq 81$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $a, b, c > 0, \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 12$  we give:  $a = x^2, b = y^2, c = z^2$*

*Hence we have:  $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = xy + yz + zx = 12$  and*



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$$\frac{(a+b+\sqrt{ab})^3}{(a+b)^2} + \frac{(b+c+\sqrt{bc})^3}{(b+c)^2} + \frac{(c+a+\sqrt{ca})^3}{(c+a)^2} = \\ = \frac{(x^2+y^2+xy)^3}{(x^2+y^2)^2} + \frac{(y^2+z^2+yz)^3}{(y^2+z^2)^2} + \frac{(z^2+x^2+zx)^3}{(z^2+x^2)^2} \geq 81 = \frac{27}{4} \underbrace{(xy+yz+zx)}_{12}$$

*Let's consider:*

$$\frac{(x^2+y^2+xy)^3}{(x^2+y^2)^2} \geq \frac{27}{4} xy \Leftrightarrow (x^2+y^2+xy)^3 \geq \frac{27}{4} xy(x^2+y^2)^2$$

$$4(x^6+y^6+(xy)^3) + 12(x^4y^2+x^5y+x^2y^4+xy^5+x^4y^2+x^2y^4) + 24x^3y^3 \\ \geq 27(x^5y+xy^5) + 24(xy)^3$$

$$4(x^6+y^6) + 24(x^4y^2+x^2y^4) \geq 26(xy)^3 + 15(x^5y+xy^5)$$

$$4[x^5(x-y)-y^5(x-y)] + 13[x^3y^2(x-y)-x^2y^3(x-y)] \\ \geq 11[x^4y(x-y)-xy^4(x-y)]$$

$$4(x-y)^2(x^4+x^3y+x^2y^2+xy^3+y^4) + 13(x-y)^2(xy)^2 \\ \geq 11(x-y)^2(x^2+xy+y^2)$$

$$4(x^4+x^3y+x^2y^2+xy^3+y^4) \geq 11(x^2+xy+y^2); x \neq y$$

$$4(x^4+y^4) + 6(xy)^2 \geq 7(x^3y+xy^3) \text{ true. Then}$$

$$\frac{(y^2+z^2+yz)^3}{(y^2+z^2)^2} \geq \frac{27}{4} yz; \frac{(z^2+x^2+zx)^3}{(z^2+x^2)^2} \geq \frac{27}{4} zx$$

$$\frac{(x^2+y^2+xy)^3}{(x^2+y^2)^2} + \frac{(y^2+z^2+yz)^3}{(y^2+z^2)^2} + \frac{(z^2+x^2+zx)^3}{(z^2+x^2)^2} \geq 81 = \frac{27}{4} \underbrace{(xy+yz+zx)}_{12}$$

$$\frac{(a+b+\sqrt{ab})^3}{(a+b)^2} + \frac{(b+c+\sqrt{bc})^3}{(b+c)^2} + \frac{(c+a+\sqrt{ca})^3}{(c+a)^2} \geq 81$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\frac{(a+b+\sqrt{ab})^3}{(a+b)^2} + \frac{(b+c+\sqrt{bc})^3}{(b+c)^2} + \frac{(c+a+\sqrt{ca})^3}{(c+a)^2} \stackrel{\text{Radon}}{\geq} \\ \geq \frac{(2a+2b+2c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}{(2a+2b+2c)^2} = \frac{(2a+2b+2c+12)^3}{(2a+2b+2c)^2} =$$



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$$= \frac{8(a+b+c+6)^3}{4(a+b+c)^2} = \frac{2(a+b+c+6)^3}{(a+b+c)^2}$$

*Let:  $t = a + b + c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 12$*

*We need to prove:*

$$\frac{2(a+b+c+6)^3}{(a+b+c)^2} \geq 81 \Leftrightarrow \frac{2(t+6)^3}{t^2} \geq 81 \Leftrightarrow 2(t+6)^3 \geq 81t^2 \Leftrightarrow$$

$$2(t+6)^3 - 81t^2 \geq 0 \Leftrightarrow 2t^3 - 45t^2 + 216t + 432 \geq 0 \Leftrightarrow (t-12)^2(2t+3) \geq 0$$

*Which is true because:  $t \geq 12$ . Proved*

**650. If  $a, b \geq 0$  then:**

$$\frac{(a+b)^3}{8} + \frac{8a^3b^3}{(a+b)^3} \geq ab\sqrt{ab} + \left( \frac{(\sqrt{a} - \sqrt{b})^2}{2} + \frac{2ab}{a+b} \right)^3$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by George Florin Șerban-Romania**

$$x = \frac{a+b}{2} = M_a; y = \frac{2ab}{a+b} = M_h; z = \sqrt{ab} = M_g \Rightarrow$$

$$x^3 + y^3 \geq z^3 + \left( \frac{a+b}{2} - \sqrt{ab} + \frac{2ab}{a+b} \right)^3$$

$$x^3 + y^3 \geq z^3 + (x - z + y)^3$$

$$(x+y)(x^2 - xy + y^2) \geq$$

$$\geq (z+x-z+y)[z^2 - z(x-z+y) + (x-z+y)^2]$$

$$(x+y)(x^2 - xy + y^2) - (z+x-z+y)[z^2 - z(x-z+y) + (x-z+y)^2] \geq 0 \Leftrightarrow$$

$$(x+y)[x^2 - xy + y^2 - z^2 + z(x-z+y) + (x-z+y)^2] \geq 0; x, y > 0 \Rightarrow x+y > 0$$

$$x^2 - xy + y^2 - z^2 + z(x-z+y) + (x-z+y)^2 = (z-x)(y-z) \geq 0 \text{ true by}$$

$$M_h = y \leq M_g = z \leq M_a = x.$$

**Solution 2 by Ravi Prakash-New Delhi-India**

Let  $\frac{1}{2}(a+b) = A; \sqrt{ab} = G; \frac{2ab}{a+b} = H$  then



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$$A + H = \frac{1}{2}(a + b) + \frac{2ab}{a + b} \geq 2 \sqrt{\frac{a + b}{2} \cdot \frac{2ab}{a + b}} = 2G \Rightarrow A + H - 2G \geq 0$$

$$\text{Also, } AH = G^2$$

*Now,*

$$\begin{aligned} \frac{(a+b)^3}{8} + \frac{8a^3b^3}{(a+b)^3} - ab\sqrt{ab} - \left( \frac{(\sqrt{a} - \sqrt{b})^2}{2} + \frac{2ab}{a+b} \right)^3 &= \\ &= A^3 + H^3 - G^3 - (A + H - G)^3 = \\ &= A^3 + H^3 - G^3 - [(A + H)^3 - 3(A + H)^2G + 3(A + H)G^2 - G^3] = \\ &= A^3 + H^3 - G^3 - \\ &- [A^3 + H^3 - G^3 + 3A^2H + 3AG^2 - 3(A + H)^2G + 3(A + H)G^2] = \\ &= 3(A + H)^2G - 3(A + H)G^2 - 3AH(A + H) = \\ &= 3(A + H)G[A + H - G - G] \geq 0 \end{aligned}$$

$$\frac{(a+b)^3}{8} + \frac{8a^3b^3}{(a+b)^3} \geq ab\sqrt{ab} + \left( \frac{(\sqrt{a} - \sqrt{b})^2}{2} + \frac{2ab}{a+b} \right)^3$$

**Solution 3 by Daoudi Abdessattar-Sbiba-Tunisia**

$$\text{Let: } u = \frac{a+b}{2} = \frac{2ab}{a+b} \text{ and } v = \frac{a+b}{2} = \sqrt{ab} \Rightarrow 0 < u \stackrel{(1)}{\leq} v \leq 1 \text{ and}$$

$$1 + u \geq 2v \Rightarrow 0 < u \leq v \leq 1 - v + u$$

$$\frac{(a+b)^3}{8} + \frac{8a^3b^3}{(a+b)^3} \geq ab\sqrt{ab} + \left( \frac{(\sqrt{a} - \sqrt{b})^2}{2} + \frac{2ab}{a+b} \right)^3$$

$$\Leftrightarrow 1 + u^3 \geq v^3 + (1 - v + u)^3 \Leftrightarrow 1 + u^3 - v^3 - (1 - v + u)^3 \geq 0$$

$$\Leftrightarrow [1^3 - (1 - v + u)^3] - (v^3 - u^3) \geq 0 \Leftrightarrow \exists \alpha \in [u, v] \text{ and}$$

$$\theta \in [1 - v + u, 1] \Rightarrow \alpha \leq \theta; (2) \text{ such that:}$$

$$LHS = 3(v - u)(\theta^2 + \alpha^2) \stackrel{(1)+(2)}{\geq} 0. \text{Proved.}$$



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**651. If  $a, b, c > 0, a + b + c = 6, 0 \leq \mu \leq 4$  then:**

$$\frac{a^2}{\mu + a^2} + \frac{b^2}{\mu + b^2} + \frac{c^2}{\mu + c^2} \leq \frac{12}{\mu + 4}$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

$$\text{For } 0 \leq \mu \leq 4; \sum_{\text{cyc}} \frac{a^2}{\mu + a^2} \leq \frac{12}{\mu + 4} \Leftrightarrow \sum_{\text{cyc}} \frac{1}{\mu + a^2} \geq \frac{3}{4 + \mu}; (1)$$

We show that (1) is true.

$$6^2 = (a + b + c)^2 \geq 3(ab + bc + ca) \Rightarrow ab + bc + ca < 12 \Rightarrow \exists \alpha \geq c \geq 0 \text{ such that}$$

$$ab + b\alpha + \alpha a = 12.$$

From  $\alpha \geq c \Rightarrow \frac{1}{\mu + c^2} \geq \frac{1}{\mu + \alpha^2}$ . So, we need to prove:

$$\frac{1}{\mu + a^2} + \frac{1}{\mu + b^2} + \frac{1}{\mu + \alpha^2} \geq \frac{3}{4 + \mu}; (2)$$

In fact, without loss of generality, assume that:  $a = \min\{a, b, \alpha\} \Rightarrow$

$$(a - b)(a - \alpha) \geq 0 \Rightarrow a^2 + b\alpha \geq ab + a\alpha + b\alpha = 12$$

Other,  $a^2 = a \cdot a \leq b \cdot \alpha \Rightarrow 12 \leq 3b\alpha \Rightarrow 4 \leq b\alpha$ .

From the CBS inequality:  $(ab + b\alpha + a\alpha) \left( \frac{1}{ab} + \frac{1}{b\alpha} + \frac{1}{a\alpha} \right) \geq 9 \Leftrightarrow 12 \left( \frac{1}{ab} + \frac{1}{b\alpha} + \frac{1}{a\alpha} \right) \geq 9$

$$\Leftrightarrow a + b + \alpha \geq \frac{3}{4} ab\alpha; (*). \text{ Now,}$$

$$\begin{aligned} \frac{1}{\mu + b^2} + \frac{1}{\mu + \alpha^2} - \frac{2}{b\alpha + \mu} &= \frac{b\alpha - b^2}{(b^2 + \mu)(b\alpha + \mu)} + \frac{b\alpha - \alpha^2}{(a^2 + \mu)(b\alpha + \mu)} = \\ &= \frac{b(\alpha - b)}{(b^2 + \mu)(b\alpha + \mu)} - \frac{\alpha(\alpha - b)}{(a^2 + \mu)(b\alpha + \mu)} = \frac{(\alpha - b)(b\alpha^2 + b\mu - \alpha b^2 - \alpha\mu)}{(a^2 + \mu)(b^2 + \mu)(b\alpha + \mu)} = \\ &= \frac{(\alpha - b)^2(b\alpha - \mu)}{(a^2 + \mu)(b^2 + \mu)(b\alpha + \mu)} \stackrel{(3)}{\geq} 0; (\text{because } (\alpha - b)^2 \geq 0, ab \geq 4 \geq \mu) \end{aligned}$$

$$\begin{aligned} \frac{1}{a^2 + \mu} + \frac{2}{b\alpha + \mu} - \frac{3}{4 + \mu} &= \frac{(4 + \mu)(b\alpha + \mu) + 2(a^2 + \mu)(4 + \mu) - 3(a^2 + \mu)(b\alpha + \mu)}{(4 + \mu)(a^2 + \mu)(b\alpha + \mu)} \\ &= \frac{(4b\alpha + 4\mu + b\alpha\mu + \mu^2) + 2(4a^2 + a^2\mu + 4\mu + \mu^2) - 3(baa^2 + \mu a^2 + b\alpha\mu + \mu^2)}{(4 + \mu)(a^2 + \mu)(b\alpha + \mu)} \end{aligned}$$



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$$\begin{aligned}
 &= \frac{4ba + 12\mu + 8a^2 - 2ba\mu - 3baa^2 - \mu a^2}{(4 + \mu)(a^2 + \mu)(ba + \mu)} = \\
 &= \frac{\mu(12 - 2ba - a^2) + (4ba + 8a^2 - 3baa^2)}{(4 + \mu)(a^2 + \mu)(ba + \mu)} \stackrel{12 - 2ba - a^2 \leq 0}{\stackrel{0 \leq \mu \leq 4}{=}} \\
 &= \frac{4(12 - 2ba - a^2) + (4ba + 8a^2 - ba^2)}{(4 + \mu)(a^2 + \mu)(ba + \mu)} \stackrel{ab + bc + ca = 12}{=} \\
 &= \frac{4a(a + b + \alpha - \frac{3}{4}aba)}{(4 + \mu)(a^2 + \mu)(ba + \mu)} \stackrel{(4)}{\geq} 0; (\text{from } (*))
 \end{aligned}$$

*From (3),(4) result (2) is true then (1) is true. Proved.*

**652. If  $a, b, c > 0, a + b + c = 3, \mu \geq \frac{3}{2}$  then:**

$$\sum_{cyc} \frac{a^2}{2(2\mu - 1)b + \sqrt[3]{4(1 + b^6)}} \geq \frac{3}{4\mu}$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

*For  $x, y > 0$  we have  $4(x^6 + y^6) \leq (3x^2 - 4xy + 3y^2)^3 \Leftrightarrow$*   
 $23x^6 - 108x^5y + 225x^4y^2 - 280x^3y^3 + 225x^2y^4 - 108xy^5 + 23y^6 \geq 0 \Leftrightarrow$   
 $(x - y)^4(23x^2 - 16xy + 23y^2) \geq 0$  which result from  $(x - y)^4 \geq 0$  equality for  $x = y$   
*and  $23x^2 - 16xy + 23y^2 > 0$  true from  $\Delta = 16^2 - 4 \cdot 23^2 < 0$ .*

*For  $x = a, y = 1$  we get  $4(a^6 + 1) \leq (3a^2 - 4a + 3)^3$  then*

$$\sqrt[3]{4(b^6 + 1)} \leq 3a^2 - 4a + 3$$

*Equality for  $a = 1$ .*

*Similarly:  $4(b^6 + 1) \leq (3b^2 - 4b + 3)^3$  and  $4(c^6 + 1) \leq (3c^2 - 4c + 3)^3 \Rightarrow$*

$$\begin{aligned}
 \Omega &= \sum_{cyc} \frac{a^2}{2(2\mu - 1)b + \sqrt[3]{4(1 + b^6)}} \geq \sum_{cyc} \frac{a^2}{3b^2 - 4b + 3 + (4\mu - 2)b} = \\
 &= \sum_{cyc} \frac{a^2}{3b^2 + (4\mu - 6)b + 3} = \sum_{cyc} \frac{(a^2)^2}{3a^2b^2 + (4\mu - 6)a^2b + 3a^2} \stackrel{\text{Bergstrom}}{\geq}
 \end{aligned}$$



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$$\geq \frac{(a^2 + b^2 + c^2)^2}{3(a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2) + (4\mu - 6)(ba^2 + cb^2 + ac^2)} \stackrel{(*)}{\geq} \frac{3}{4\mu}$$

$$(*) \Leftrightarrow 4\mu \left( \sum a^4 + 2 \sum a^2b^2 \right) \geq$$

$$\geq 3 \left( 3 \sum a^2b^2 + 3 \sum a^2 + (4\mu - 6)(ba^2 + cb^2 + ac^2) \right)$$

$$4\mu \sum a^4 + 8\mu \sum a^2b^2 \geq 9 \sum a^2b^2 + 9 \sum a^2 + 3(4\mu - 6)(ba^2 + cb^2 + ac^2)$$

$$4\mu \sum a^4 + (8\mu - 9) \sum a^2b^2 \geq 3 \sum a^2 + 3(4\mu - 6)(ba^2 + cb^2 + ac^2)$$

$$(4\mu - 6) \sum a^4 + (8\mu - 12) \sum a^2b^2 + [4 \sum a^4 + 3 \sum a^2b^2]$$

$$\geq 3 \sum a^2 + 3(4\mu - 6)(ba^2 + cb^2 + ac^2)$$

$$(4\mu - 6) \left( \sum a^4 + 2 \sum a^2b^2 - 3(ba^2 + cb^2 + ac^2) \right)$$

$$+ 3 \left( 2 \sum a^4 + \sum a^2b^2 - 3 \sum a^2 \right) \stackrel{(**)}{\geq} 0$$

*With  $a + b + c = 3, \mu \geq \frac{3}{2} \Rightarrow 4\mu - 6 \geq 0$  we have:*

$$\begin{aligned} & \sum a^4 + 2 \sum a^2b^2 - 3(ba^2 + cb^2 + ac^2) = \\ &= \sum a^4 + 2 \sum a^2b^2 - (a + b + c)(ba^2 + cb^2 + ac^2) = \sum a^4 + 2 \sum a^2b^2 - \\ & \quad - [\sum a^2b^2 + abc(a + b + c) + (ba^3 + cb^3 + ac^3)] = \\ &= \sum a^4 + \sum a^2b^2 - [abc(a + b + c) + (ba^3 + cb^3 + ac^3)] = \\ &= [\sum a^4 - (ba^3 + cb^3 + ac^3)] + [\sum a^2b^2 - abc(a + b + c)] \stackrel{(1)}{\geq} 0 \end{aligned}$$

*We have (1) is true, because:*

$$a^4 + a^4 + a^4 + b^4 \stackrel{AM-GM}{\geq} 4\sqrt[4]{(a^3b)^4} = 4ba^3$$

$$b^4 + b^4 + b^4 + c^4 \stackrel{AM-GM}{\geq} 4\sqrt[4]{(b^3c)^4} = 4cb^3$$

$$c^4 + c^4 + c^4 + a^4 \stackrel{AM-GM}{\geq} 4\sqrt[4]{(c^3a)^4} = 4ac^3$$



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$$\Rightarrow 4(a^4 + b^4 + c^4) \geq 4(ba^3 + cb^3 + ac^3) \Rightarrow \sum a^4 - (ba^3 + cb^3 + ac^3) \stackrel{(2)}{\geq} 0$$

$$a^2b^2 + b^2c^2 + c^2a^2 \stackrel{AM-GM}{\geq} abc(a+b+c) \Rightarrow \sum a^2b^2 - abc(a+b+c) \stackrel{(3)}{\geq} 0$$

*From (1),(2)  $\Rightarrow$  (1) is true.*

$$3 \sum a^2 \stackrel{\Sigma a=3}{=} \frac{(\sum a)^2}{3} \cdot \sum a^2 \stackrel{BCS}{\leq} \sum a^2 \cdot \sum a^2 = (\sum a^2)^2 \stackrel{(4)}{\leq} 2 \sum a^4 + \sum a^2b^2$$

$$(4) \Leftrightarrow \sum a^4 \geq \sum a^2b^2 \text{ (true by } x^2 + y^2 + z^2 \geq xy + yz + zx)$$

$$2 \sum a^4 + \sum a^2b^2 \geq 3 \sum a^2 \Rightarrow$$

$$3(2 \sum a^4 + \sum a^2b^2 - 3 \sum a^2) \stackrel{(5)}{\geq} 0$$

*From (4),(5) and  $4\mu - 6 \geq 0 \Rightarrow (**)\text{true} \Rightarrow (*)\text{true}.$*

**653. If  $x, y, z > 0, xy + yz + zx = 3, \mu \geq \frac{13}{27}$  then:**

$$\mu(x^2 + y^2 + z^2) + x^2y^2z^2 \geq 3\mu + 1$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

$x, y, z > 0, xy + yz + zx = 3 \Leftrightarrow \frac{x}{\sqrt{3}} \cdot \frac{y}{\sqrt{3}} + \frac{y}{\sqrt{3}} \cdot \frac{z}{\sqrt{3}} + \frac{z}{\sqrt{3}} \cdot \frac{x}{\sqrt{3}} = 1 \Rightarrow (\exists) \Delta ABC \text{ such that:}$

$$\frac{x}{\sqrt{3}} = \tan \frac{A}{2}; \frac{y}{\sqrt{3}} = \tan \frac{B}{2}; \frac{z}{\sqrt{3}} = \tan \frac{C}{2} \Rightarrow x = \sqrt{3} \tan \frac{A}{2}; y = \sqrt{3} \tan \frac{B}{2}; z = \sqrt{3} \tan \frac{C}{2}$$

*Hence,  $\mu(x^2 + y^2 + z^2) + x^2y^2z^2 \geq 3\mu + 1; \mu \geq \frac{13}{27} \Leftrightarrow$*

$$3\mu \sum_{cyc} \tan^2 \frac{A}{2} + 27 \left( \prod_{cyc} \tan \frac{A}{2} \right)^2 \geq 3\mu + 1 \Leftrightarrow$$

$$3\mu \cdot \frac{(4R+r)^2 - 2s^2}{s^2} + 27 \left( \frac{r}{s} \right)^2 \geq 3\mu + 1 \Leftrightarrow$$

$$3\mu[(4R+r)^2 - 2s^2] + 27r^2 \geq (3\mu + 1)s^2 \Leftrightarrow$$

$$3\mu(4R+r)^2 + 27r^2 \stackrel{(*)}{\geq} (9\mu + 1)s^2$$

$$\text{But: } s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}$$



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*Let:  $t = \frac{R}{r} \geq 2$ , we must show that:*

$$3\mu(4t+1)^2 + 27 \geq (9\mu+1) \left[ (2t^2 + 10t - 1) + (2t-4)\sqrt{t^2 - 2t} \right] \Leftrightarrow$$

$$3\mu(4t+1)^2 + 27 - (9\mu+1)(2t^2 + 10t - 1) \geq (9\mu+1)(2t-4)\sqrt{t^2 - 2t} \Leftrightarrow$$

$$3\mu(16t^2 + 8t + 1) + 27 - (9\mu+1)(2t^2 + 10t - 1) \geq (9\mu+1)(2t-4)\sqrt{t^2 - 2t} \Leftrightarrow$$

$$2(t-2)[(15\mu-1)t - 3\mu - 7] \geq 2(9\mu+1)(t-2)\sqrt{t^2 - 2t} \Leftrightarrow$$

$$(t-2)[(15\mu-1)t - 3\mu - 7] \geq (9\mu+1)(t-2)\sqrt{t^2 - 2t}$$

*Because:  $t \geq 2 \Rightarrow t-2 \geq 0$*

*We need to prove:*

$$(15\mu-1)t - 3\mu - 7 \geq (9\mu+1)(t-2)\sqrt{t^2 - 2t} \Leftrightarrow$$

$$[(15\mu-1)t - 3\mu - 7]^2 \geq (9\mu+1)^2(t^2 - 2t)$$

$$(15\mu-1)t - 3\mu - 7 \stackrel{t \geq 2}{\geq} (15\mu-1) \cdot 2 - 3\mu - 7 = 27\mu - 9 \stackrel{\mu \geq \frac{13}{27}}{\geq} 13 - 9 = 4 > 0 \Leftrightarrow$$

$$(144\mu^2 - 48\mu)t^2 + (72\mu^2 - 168\mu + 16)t + 9\mu^2 + 49 \stackrel{(**)}{\geq} 0 \Leftrightarrow$$

$$(144\mu^2 - 48\mu)t^2 + 8(9\mu^2 - 21\mu + 2)t + 9\mu^2 + 49 \stackrel{(**)}{\geq} 0$$

*Which is clearly true, because:*

$$\Delta_t = -64(9\mu-4)(9\mu+1)^2 \leq -\frac{16384}{27} < 0; \left( \text{for } \mu \geq \frac{13}{27} \right)$$

$$a = 144\mu^2 - 48\mu = 48\pi(3\mu-1) \stackrel{\mu \geq \frac{13}{27}}{\geq} 0 \Rightarrow (**)\text{true} \Rightarrow (*)\text{true}.$$

**654. If  $a, b, c > 0$ ;  $a + b + c = 1$  then prove:**

$$\frac{9}{136} \leq \frac{a^2}{a^3 + 5} + \frac{b^2}{b^3 + 5} + \frac{c^2}{c^3 + 5} \leq \frac{1}{6}$$

*Proposed by Jalil Hajimir-Toronto-Canada*

*Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam*

*1) Prove that the inequality:*



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$$\frac{a^2}{a^3 + 5} + \frac{b^2}{b^3 + 5} + \frac{c^2}{c^3 + 5} \geq \frac{9}{136}; \quad (1)$$

**Put:**  $f(x) = \frac{x^2}{x^3 + 5}$ ;  $x \in (0, 1)$ ;  $f'(x) = \frac{x(10 - x^3)}{(x^3 + 5)^2} > 0$  since  $\begin{cases} x(10 - x^3) > 0 \\ (x^3 + 5)^2 > 0 \end{cases}$ ;  $(x \in (0, 1)) \Rightarrow f -\text{increasing.}$

**Case 1.** One of three number  $a, b, c$  isn't less than  $\sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}$ . WLOG  $a \geq$

$\sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}$  and since  $f -\text{increasing result:}$

$$f(a) \geq f\left(\sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}\right) > \frac{9}{136}$$

On the other hand, we have  $f(b) > 0; f(c) > 0 \Rightarrow f(a) + f(b) + f(c) > \frac{9}{136}$  which

$$\frac{a^2}{a^3 + 5} + \frac{b^2}{b^3 + 5} + \frac{c^2}{c^3 + 5} \geq \frac{9}{136}$$

**Case 2.** Both three number  $a, b, c$  less than  $\sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}$ , which we have

$$a < \sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}; b < \sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}; c < \sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}$$

We have:

$$f''(x) = \frac{2(x^6 - 35x^3 + 25)}{(x^3 + 5)^3} = \frac{2\left(x^3 - \frac{5}{2}(7 - 3\sqrt{5})\right)\left(x^3 - \frac{5}{2}(7 + 3\sqrt{5})\right)}{(x^3 + 5)^3} > 0$$

$$\forall x \in \left(0, \sqrt[3]{\frac{5}{2}(7 - 3\sqrt{5})}\right) \text{ since } \begin{cases} x^3 - \frac{5}{2}(7 - 3\sqrt{5}) < 0 \\ x^3 - \frac{5}{2}(7 + 3\sqrt{5}) < 0 \end{cases}$$

So  $f -\text{convex, by Jensen Inequality, we have:}$

$$f(a) + f(b) + f(c) \geq f\left(\frac{a+b+c}{3}\right) = 3f\left(\frac{1}{3}\right)$$

$$\frac{a^2}{a^3 + 5} + \frac{b^2}{b^3 + 5} + \frac{c^2}{c^3 + 5} \geq \frac{9}{136}$$



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*The inequality occurs when  $a = b = c = \frac{1}{3}$*

**2) Prove the inequality:**

$$\frac{a^2}{a^3 + 5} + \frac{b^2}{b^3 + 5} + \frac{c^2}{c^3 + 5} \leq \frac{1}{6}; \quad (2)$$

*By AM-GM inequality, we have:*

$$x^3 + 5 = x^3 + 1 + 1 + 3 \geq 3x + 3; \forall x > 0$$

*On the other hand, we have:  $(x - 1)(2x + 1) < 0 \Rightarrow 2x^2 - x - 1 < 0 \Rightarrow x + 1 >$*

$$2x^2; \forall x < 1$$

$$\text{So, } x^3 + 5 > 6x^2 \Leftrightarrow \frac{x^2}{x^3 + 5} < \frac{x}{6}; \forall x \in (0, 1) \Rightarrow$$

$$\frac{a^2}{a^3 + 5} + \frac{b^2}{b^3 + 5} + \frac{c^2}{c^3 + 5} < \frac{a + b + c}{6} \Rightarrow \frac{a^2}{a^3 + 5} + \frac{b^2}{b^3 + 5} + \frac{c^2}{c^3 + 5} \leq \frac{1}{6}$$

*From (1),(2) we have the thing to prove.*

**655. If  $x, y, z > 0$  then:**

$$\frac{1}{\sqrt{(x+y)(y+z)}} + \frac{1}{\sqrt{(y+z)(z+x)}} + \frac{1}{\sqrt{(z+x)(x+y)}} \leq \frac{3}{2} \sqrt{\frac{3}{xy + yz + zx}}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam**

$$\begin{aligned} \frac{1}{\sqrt{(x+y)(y+z)}} + \frac{1}{\sqrt{(y+z)(z+x)}} + \frac{1}{\sqrt{(z+x)(x+y)}} &\stackrel{BCS}{\leq} \\ &\leq \sqrt{3 \left( \frac{1}{(x+y)(y+z)} + \frac{1}{(y+z)(z+x)} + \frac{1}{(z+x)(x+y)} \right)} \\ \frac{1}{\sqrt{(x+y)(y+z)}} + \frac{1}{\sqrt{(y+z)(z+x)}} + \frac{1}{\sqrt{(z+x)(x+y)}} &\leq \sqrt{\frac{6(x+y+z)}{(x+y)(y+z)(z+x)}}; \quad (1) \end{aligned}$$

*On the other hand, we have:*

$$9(x+y)(y+z)(z+x) = 9(x+y+z)(xy+yz+zx) - 9xyz$$

*And by Cauchy inequality, we have:*

$$9xyz = 3\sqrt[3]{xyz} \cdot 3\sqrt[3]{x^2y^2z^2} \leq (x+y+z)(xy+yz+zx)$$



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*So*

$$9(x+y)(y+z)(z+x) \geq 9(x+y+z)(xy+yz+zx) - (x+y+z)(xy+yz+zx)$$

*Or we have:*

$$9(x+y)(y+z)(z+x) \geq 8(x+y+z)(xy+yz+zx) \Rightarrow$$

$$\frac{6(x+y+z)}{(x+y)(y+z)(z+x)} \leq \frac{27}{4(xy+yz+zx)} \Leftrightarrow$$

$$\sqrt{\frac{6(x+y+z)}{(x+y)(y+z)(z+x)}} \leq \frac{3}{2} \sqrt{\frac{3}{xy+yz+zx}}; \quad (2)$$

*From (11),(2) we have the thing to prove:*

$$\frac{1}{\sqrt{(x+y)(y+z)}} + \frac{1}{\sqrt{(y+z)(z+x)}} + \frac{1}{\sqrt{(z+x)(x+y)}} \leq \frac{3}{2} \sqrt{\frac{3}{xy+yz+zx}}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

$$\begin{aligned} & \frac{1}{\sqrt{(x+y)(y+z)}} + \frac{1}{\sqrt{(y+z)(z+x)}} + \frac{1}{\sqrt{(z+x)(x+y)}} = \\ &= \frac{\sqrt{z+x}}{\sqrt{(x+y)(y+z)(z+x)}} + \frac{\sqrt{x+y}}{\sqrt{(x+y)(y+z)(z+x)}} + \frac{\sqrt{y+z}}{\sqrt{(z+x)(x+y)(y+z)}} \leq \\ & \leq \frac{3}{2} \sqrt{\frac{3}{xy+yz+zx}} \end{aligned}$$

$$\text{Iff } \sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} \leq \frac{3\sqrt{3}}{2} \cdot \sqrt{\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}}$$

$$\sqrt{6(x+y+z)} \leq \frac{3\sqrt{3}}{2} \cdot \sqrt{\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}}$$

$$6(x+y+z)(xy+yz+zx) \leq \frac{27}{4}(x+y)(y+z)(z+x)$$

$$8(x+y+z)(xy+yz+zx) \leq 9(x+y)(y+z)(z+x)$$

$$8(x^2y + y^2z + z^2x + x^2z + z^2y + y^2x + 3xyz) \leq$$

$$\leq 9(x^2y + y^2z + z^2x + x^2z + z^2y + y^2x + 2xyz)$$



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**Solution 3 by Daoudi Abdessatar-Sbiba-Tunisia**

$$x, y, z > 0 \Rightarrow \sum \sqrt{\frac{1}{(x+y)(x+z)}} \leq \frac{3}{2} \sqrt{\frac{3}{xy+yz+zx}}; (I)$$

$$\text{Let } p = \sum x, q = \sum xy \Rightarrow p^2 \geq 3q; (1)$$

$$(I) \Leftrightarrow \sum \sqrt{\frac{q}{x^2+q}} \leq \frac{3\sqrt{3}}{2} \Leftrightarrow \sum \sqrt{\frac{q}{x^2+q} \cdot \frac{3}{4}} \leq \frac{9}{4}$$

$$\sum \sqrt{\frac{q}{x^2+q} \cdot \frac{3}{4}} \stackrel{AM-GM}{\leq} \frac{1}{2} \cdot \sum \frac{q}{x^2+q} + \frac{9}{8} = \frac{21}{8} - \frac{1}{2} \cdot \sum \frac{x^2}{x^2+q} \stackrel{(2)}{\leq} \frac{9}{4}$$

$$(2) \Leftrightarrow \sum \frac{x^2}{x^2+q} \geq \frac{3}{4}$$

*True from:*

$$\sum \frac{x^2}{x^2+q} \stackrel{CBS}{\geq} \frac{p^2}{p^2+q} \stackrel{(1)}{\geq} \frac{p^2}{p^2+\frac{p^2}{3}} = \frac{3}{4}$$

**656. If  $a, b > 0$  then:**

$$\left( \frac{2\sqrt{ab}}{a+b} + \frac{a+b}{2\sqrt{ab}} \right) \left( \sqrt{\frac{a^2+b^2}{2ab}} + \sqrt{\frac{2ab}{a^2+b^2}} \right) \left( \frac{\sqrt{2(a^2+b^2)}}{a+b} + \frac{a+b}{\sqrt{2(a^2+b^2)}} \right) \leq \left( \frac{a}{b} + \frac{b}{a} \right)^3$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

Let  $Q = \sqrt{\frac{a^2+b^2}{2}}, A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$  and of course :  $Q \geq A \geq G \geq H$

$$\begin{aligned} \frac{2\sqrt{ab}}{a+b} + \frac{a+b}{2\sqrt{ab}} + 2 &= \frac{G}{A} + \frac{A}{G} + 2 \leq \frac{a}{b} + \frac{b}{a} + 2 \Leftrightarrow \frac{(A+G)^2}{AG} \leq \frac{(a+b)^2}{ab} = \frac{4A^2}{G^2} \\ &\stackrel{(1)}{\Leftrightarrow} 4A^3 \geq G(A+G)^2 \end{aligned}$$

$$\begin{aligned} \because A \geq G \therefore 2A \geq A+G &\Rightarrow 4A^2 \geq (A+G)^2 \text{ and combined with } A \geq G, \text{ we get : } 4A^3 \\ &\geq G(A+G)^2 \Rightarrow (1) \text{ is true} \end{aligned}$$



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$$\therefore \frac{2\sqrt{ab}}{a+b} + \frac{a+b}{2\sqrt{ab}} \stackrel{(i)}{\geq} \frac{a}{b} + \frac{b}{a}$$

$$Again, \sqrt{\frac{a^2+b^2}{2ab}} + \sqrt{\frac{2ab}{a^2+b^2}} + 2 = \frac{Q}{G} + \frac{G}{Q} + 2 \leq \frac{a}{b} + \frac{b}{a} + 2 \Leftrightarrow \frac{(Q+G)^2}{QG} \leq \frac{(a+b)^2}{ab}$$

$$= \frac{4A^2}{G^2} \Leftrightarrow 4A^2Q \stackrel{(2)}{\geq} G(Q+G)^2$$

$$\begin{aligned} a+b &\geq \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} \Leftrightarrow a+b - \sqrt{ab} \geq \sqrt{\frac{a^2+b^2}{2}} \Leftrightarrow (a+b)^2 + ab - 2\sqrt{ab}(a+b) \\ &\geq \frac{a^2+b^2}{2} \end{aligned}$$

$$\begin{aligned} \Leftrightarrow a^2 + b^2 + 6ab - 4\sqrt{ab}(a+b) &\geq 0 \Leftrightarrow (a+b)^2 - 4\sqrt{ab}(a+b) + 4ab \geq 0 \\ \Leftrightarrow (a+b - 2\sqrt{ab})^2 &\geq 0 \rightarrow true \end{aligned}$$

$$\begin{aligned} \therefore a+b &\geq \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} \Rightarrow 2A \geq Q+G \Rightarrow 4A^2 \geq (Q+G)^2 and combined with Q \\ &\geq G, we get : 4A^2Q &\geq G(Q+G)^2 \Rightarrow (2) is true \end{aligned}$$

$$\therefore \sqrt{\frac{a^2+b^2}{2ab}} + \sqrt{\frac{2ab}{a^2+b^2}} \stackrel{(ii)}{\geq} \frac{a}{b} + \frac{b}{a}$$

$$Also, \frac{\sqrt{2(a^2+b^2)}}{a+b} + \frac{a+b}{\sqrt{2(a^2+b^2)}} = \frac{Q}{A} + \frac{A}{Q} = \frac{Q^2+A^2}{AQ} \leq \frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} = \frac{2Q^2}{G^2}$$

$$\Leftrightarrow \frac{2Q^2}{Q^2+A^2} \geq \frac{G^2}{AQ} \Leftrightarrow \frac{Q^2-A^2}{Q^2+A^2} \stackrel{(3)}{\geq} \frac{G^2-AQ}{AQ}$$

$$\because Q \geq A \therefore \frac{Q^2-A^2}{Q^2+A^2} \geq 0 and \because G^2 = G, G \leq A, Q \therefore \frac{G^2-AQ}{AQ} \leq 0 \Rightarrow \frac{Q^2-A^2}{Q^2+A^2} \geq 0$$

$$\geq \frac{G^2-AQ}{AQ} \Rightarrow (3) is true$$

$$\therefore \frac{\sqrt{2(a^2+b^2)}}{a+b} + \frac{a+b}{\sqrt{2(a^2+b^2)}} \stackrel{(iii)}{\geq} \frac{a}{b} + \frac{b}{a}$$



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$$\begin{aligned}
 \text{(i). (ii). (iii)} \Rightarrow & \left( \frac{2\sqrt{ab}}{a+b} + \frac{a+b}{2\sqrt{ab}} \right) \left( \sqrt{\frac{a^2+b^2}{2ab}} + \sqrt{\frac{2ab}{a^2+b^2}} \right) \left( \frac{\sqrt{2(a^2+b^2)}}{a+b} \right. \\
 & \left. + \frac{a+b}{\sqrt{2(a^2+b^2)}} \right) \leq \left( \frac{a}{b} + \frac{b}{a} \right)^3 \quad (\text{Proved})
 \end{aligned}$$

**Solution 2 by Abdallah El Farissi-Algerie**

$$\text{Let } A = \frac{2\sqrt{ab}}{a+b}; B = \sqrt{\frac{a^2+b^2}{2ab}}; C = \frac{\sqrt{2(a^2+b^2)}}{a+b}$$

$$\text{We have: } 1 \leq A \leq \frac{a}{b}; 1 \leq B \leq \frac{a}{b}; 1 \leq C \leq \frac{a}{b}$$

Let  $f(x) = x + \frac{1}{x}$ ,  $f$  -is positive and increasing function in  $[1, \infty)$ , it follows that

$$f(A)f(B)f(C) \leq f^3\left(\frac{a}{b}\right) \Leftrightarrow$$

$$\left( \frac{2\sqrt{ab}}{a+b} + \frac{a+b}{2\sqrt{ab}} \right) \left( \sqrt{\frac{a^2+b^2}{2ab}} + \sqrt{\frac{2ab}{a^2+b^2}} \right) \left( \frac{\sqrt{2(a^2+b^2)}}{a+b} + \frac{a+b}{\sqrt{2(a^2+b^2)}} \right) \leq \left( \frac{a}{b} + \frac{b}{a} \right)^3$$

**657.**

$$\forall a, b, c > 0, \sqrt[15]{8192(a^5+b^5)^2(b^5+c^5)^2(c^5+a^5)^2} \leq a^2 + b^2 + c^2$$

*Proposed by Jalil Hajimir-Toronto-Canada*

**Solution by Soumava Chakraborty-Kolkata-India**

$$2(a^2 - ab + b^2)^2 - (a^4 + b^4) - (a - b)^4 = 0 \Rightarrow 2(a^2 - ab + b^2)^2 - (a^4 + b^4)$$

$$= (a - b)^4 \geq 0 \Rightarrow 2(a^2 - ab + b^2)^2 \stackrel{(i)}{\geq} a^4 + b^4$$

$$\Rightarrow 2(a^2 - ab + b^2)^2 \stackrel{(i)}{\geq} a^4 + b^4$$

$$\text{Now, } a^5 + b^5 = (a + b) \left( a^4 + b^4 - ab(a^2 - ab + b^2) \right) \stackrel{\text{by (i)}}{\leq} (a$$

$$+ b) \left( 2(a^2 - ab + b^2)^2 - ab(a^2 - ab + b^2) \right)$$

$$= (a + b)(a^2 - ab + b^2)(2a^2 - 3ab + 2b^2) \Rightarrow (a^5 + b^5)^2$$

$$\leq (a + b)^2(a^2 - ab + b^2)^2(2a^2 - 3ab + 2b^2)^2 \text{ and analogs}$$



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$$\begin{aligned}
 & \therefore \sqrt[15]{(a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2} \leq \\
 & \sqrt[15]{(a+b)^2(a^2 - ab + b^2)^2(2a^2 - 3ab + 2b^2)^2(b+c)^2(b^2 - bc + c^2)^2(2b^2 - 3bc + 2c^2)^2(c+a)^2(c^2 - ca + a^2)^2(2c^2 - 3ca + 2a^2)^2} \\
 & \stackrel{\substack{\text{weighted} \\ AM - GM}}{\geq} \frac{\sum(a+b)^2 + 2(a^2 - ab + b^2 + 2a^2 - 3ab + 2b^2 + b^2 - bc + c^2 + 2b^2 - 3bc + 2c^2 + c^2 - ca + a^2 + 2c^2 - 3ca + 2a^2)}{15} \\
 & = \frac{14a^2 + 14b^2 + 14c^2 - 6ab - 6bc - 6ca}{15} \\
 & \therefore \sqrt[15]{(a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2} \stackrel{(1)}{\geq} \frac{14 \sum a^2 - 6 \sum ab}{15} \text{ and } \therefore \sqrt[15]{8192} \\
 & = \sqrt[15]{2^{13}} \stackrel{(2)}{\geq} 1 \\
 & \therefore \sqrt[15]{8192(a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2} \stackrel{\substack{\text{by (1) and (2)} \\ ?}}{\geq} \frac{14 \sum a^2 - 6 \sum ab}{15} \stackrel{?}{\geq} \sum a^2 \\
 & \Leftrightarrow 14 \sum a^2 - 6 \sum ab \stackrel{?}{\geq} 15 \sum a^2 \\
 & \Leftrightarrow \sum a^2 + 6 \sum ab \stackrel{?}{\geq} 0 \rightarrow \text{true} \therefore \sqrt[15]{8192(a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2} \\
 & < a^2 + b^2 + c^2 \text{ (Proved)}
 \end{aligned}$$

**658. If  $x, y, z \geq 0, x^3 + y^3 + z^3 = 24$  then:**

$$(x^5y + y^5z + z^5x)(x^2y + y^2z + z^2x) \geq 576xyz$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

We have:  $576 = 24^2 = (x^3 + y^3 + z^3)^2$

Inequality becomes as:

$$(x^5y + y^5z + z^5x)(x^2y + y^2z + z^2x) \geq (x^3 + y^3 + z^3)^2 \cdot xyz; \quad (1)$$

$$x^5y + y^5z + z^5x = \frac{x^6}{y} + \frac{y^6}{z} + \frac{z^6}{x} \stackrel{\text{Bergstrom}}{\geq} \frac{(x^3 + y^3 + z^3)^2}{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}} = \frac{(x^3 + y^3 + z^3)^2 \cdot xyz}{x^2z + y^2x + z^2y}$$

WLOG, suppose:  $x \geq y \geq z$ . We need to prove:

$$x^2y + y^2z + z^2x \geq x^2z + y^2x + z^2y \Leftrightarrow (x-y)(y-z)(x-z) \geq 0$$

true by  $x \geq y \geq z \Rightarrow x-y \geq 0; x-z \geq 0; y-z \geq 0 \Rightarrow (1) \text{ is true.}$



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*Proved.*

**Solution 2 by Khaled Abd Imouti-Damascus-Syria**

$$\begin{aligned}
 & (x^5y + y^5z + z^5x)(x^2y + y^2z + z^2x) \geq 576xyz \Leftrightarrow \\
 & (x^5y + y^5z + z^5x)(x^2y + y^2z + z^2x) \geq (x^3 + y^3 + z^3)^2 \cdot xyz \Leftrightarrow \\
 & (x^5y + y^5z + z^5x) \left( \frac{x}{z} + \frac{y}{x} + \frac{z}{y} \right) \geq (x^3 + y^3 + z^3)^2
 \end{aligned}$$

*By using BCS Inequality, we have:*

$$\begin{aligned}
 & \left( x^3 \cdot \sqrt{\frac{y}{x}} \cdot \sqrt{\frac{x}{y}} + y^3 \cdot \sqrt{\frac{z}{y}} \cdot \sqrt{\frac{y}{z}} + z^3 \cdot \sqrt{\frac{x}{z}} \cdot \sqrt{\frac{z}{x}} \right)^2 \leq \left( x^6 \cdot \frac{y}{x} + y^6 \cdot \frac{z}{y} + z^6 \cdot \frac{x}{z} \right) \left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \\
 & \Leftrightarrow (x^3 + y^3 + z^3)^2 \leq (x^5y + y^5z + z^5x) \left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \\
 & \text{WLOG, suppose } x \leq y \leq z \text{ then: } \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \leq \frac{x}{z} + \frac{y}{x} + \frac{z}{y}. \text{ So,} \\
 & (x^3 + y^3 + z^3)^2 \leq (x^5y + y^5z + z^5x) \left( \frac{x}{z} + \frac{y}{x} + \frac{z}{y} \right)
 \end{aligned}$$

**659. If  $a, b, c \geq 1$  then prove:**

$$\frac{a^a}{(b+c)\left(\frac{1}{b} + \frac{1}{c} + \log(bc)\right)} + \frac{b^b}{(c+a)\left(\frac{1}{c} + \frac{1}{a} + \log(ca)\right)} + \frac{c^c}{(a+b)\left(\frac{1}{a} + \frac{1}{b} + \log(ab)\right)} \geq \frac{3}{4}$$

*Proposed by Pavlos Trifon-Greece*

**Solution by proposer**

Let be the function:  $f: [1, \infty) \rightarrow \mathbb{R}, f(x) = \frac{1}{x} + \log x, f'(x) > 0 \Rightarrow f \uparrow [1, \infty)$

Applying Chebyshev's Inequality for  $(a, b); \left(\frac{1}{a} + \log a, \frac{1}{b} + \log b\right)$  we get:

$$\begin{aligned}
 & (a+b)\left(\frac{1}{a} + \log a + \frac{1}{b} + \log b\right) \leq 2\left(a\left(\frac{1}{a} + \log a\right) + b\left(\frac{1}{b} + \log b\right)\right) = \\
 & = 2(2 + \log a^a + \log b^b) \leq 2(2 + a^a - 1 + b^b - 1) = 2(a^a + b^b) \Rightarrow \\
 & \frac{c^c}{(a+b)\left(\frac{1}{a} + \frac{1}{b} + \log(ab)\right)} \geq \frac{1}{2} \cdot \frac{c^c}{a^a + b^b}
 \end{aligned}$$

*Equality holds when  $a = b = 1$ . Similarly:*



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$$\frac{a^a}{(b+c)\left(\frac{1}{b}+\frac{1}{c}+\log(bc)\right)} \geq \frac{1}{2} \cdot \frac{a^a}{b^b+c^c}$$

$$\frac{b^b}{(c+a)\left(\frac{1}{c}+\frac{1}{a}+\log(ca)\right)} \geq \frac{1}{2} \cdot \frac{b^b}{c^c+a^a}$$

*So, we get:*

$$\sum_{cyc} \frac{a^a}{(b+c)\left(\frac{1}{b}+\frac{1}{c}+\log(bc)\right)} \geq \frac{1}{2} \left( \frac{c^c}{a^a+b^b} + \frac{b^b}{c^c+a^a} + \frac{a^a}{b^b+c^c} \right) \stackrel{\text{Nesbitt}}{\geq} \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$$

*Equality holds when  $a = b = c = 1$*

**660. If  $a, b > 0$  then:**

$$\sqrt[3]{\frac{a^3+b^3}{2}} \cdot \sqrt[4]{\frac{a^4+b^4}{2}} \cdot \sqrt[5]{\frac{a^5+b^5}{2}} \leq \frac{a^5+b^5}{a^2+b^2}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $a, b > 0$  we have:*

$$\sqrt[3]{\frac{a^3+b^3}{2}} \cdot \sqrt[4]{\frac{a^4+b^4}{2}} \cdot \sqrt[5]{\frac{a^5+b^5}{2}} = \left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}} \cdot \left(\frac{a^4+b^4}{2}\right)^{\frac{1}{4}} \cdot \left(\frac{a^5+b^5}{2}\right)^{\frac{1}{5}} =$$

$$= \left(\frac{a^3+b^3}{2}\right)^{\frac{20}{60}} \cdot \left(\frac{a^4+b^4}{2}\right)^{\frac{15}{60}} \cdot \left(\frac{a^5+b^5}{2}\right)^{\frac{12}{60}} \leq \frac{a^5+b^5}{a^2+b^2} \Leftrightarrow$$

$$(a^2+b^2)^{60} \cdot (a^3+b^3)^{20} \cdot (a^4+b^4)^{15} \cdot (a^5+b^5)^{12} = 2^{47} \cdot (a^5+b^5)^{60} \Leftrightarrow$$

$$(a^2+b^2)^{60} \cdot (a^3+b^3)^{20} \cdot (a^4+b^4)^{15} \leq 2^{47} \cdot (a^5+b^5)^{48} \Leftrightarrow$$

$$(a^2+b^2)^{60} \cdot (a^3+b^3)^{20} \cdot (a^4+b^4)^{15} \leq 2^{47} \left[ \frac{(a^2+b^2)(a^3+b^3)}{2} \right]^{20} (a^5+b^5)^{28} \Leftrightarrow$$

$$(a^2+b^2)^{40} \cdot (a^4+b^4)^{15} \leq 2^{27} \cdot (a^5+b^5)^{28} \Leftrightarrow$$

$$(a^2+b^2)^{10} \cdot (a^4+b^4)^{15} \leq 2^{27} \cdot (a^5+b^5)^{28} \Leftrightarrow$$

$$(2^2(a^4+b^4)^2)^{10} \cdot (a^4+b^4)^{15} \leq 2^{27} \cdot (a^5+b^5)^{28} \Leftrightarrow$$



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$$(a^4 + b^4)^{35} \leq 2^7 \cdot (a^5 + b^5)^{28} \Leftrightarrow$$

$$(a^4 + b^4)^5 \leq 2 \cdot (a^5 + b^5)^4 \Leftrightarrow$$

$$2 \cdot (a^5 + b^5)^4 \leq 2 \cdot (a^5 + b^5)^4 \text{ (true).}$$

**Solution 2 Pavlos Trifon-Greece**

Let  $A_n = \sqrt[n]{\frac{a^n+b^n}{2}} \Rightarrow A_3 \leq A_4 \leq A_5$  (power means).

$$A_3 \cdot A_4 \cdot A_5 \leq \frac{a^5 + b^5}{a^2 + b^2} \Leftrightarrow$$

$$A_5^3 \leq \frac{a^5 + b^5}{a^2 + b^2} \Leftrightarrow \sqrt[5]{\frac{a^5 + b^5}{2}} \leq \sqrt[3]{\frac{a^5 + b^5}{a^2 + b^2}} \Leftrightarrow$$

$$\left(\frac{a^5 + b^5}{2}\right)^3 \leq \left(\frac{a^5 + b^5}{a^2 + b^2}\right)^5 \Leftrightarrow (a^2 + b^2)^5 \leq 8(a^5 + b^5)^2 \stackrel{ab=1}{\Leftrightarrow}$$

$$\left(a^2 + \frac{1}{a^2}\right)^5 \leq 8 \left(a^5 + \frac{1}{a^5}\right)^2 \Leftrightarrow (a^4 + 1)^5 \leq 8(a^{10} + 1)^2 \Leftrightarrow$$

$$(a^4 + 1)^5 \leq (1 + 1)(1 + 1)(1 + 1)(a^{10} + 1)(a^{10} + 1) \text{ (Holder).}$$

**661. If  $x, y, z > 0$  then:**

$$\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} + \frac{xy + yz + zx}{8} \geq \frac{9}{8}$$

*Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

*For  $x, y > 0$  we have:*

$$(\sqrt{(xy)^2} + 1^2) \left( \sqrt{\left(\frac{x}{y}\right)^2 + 1^2} \right) \stackrel{BCS}{\geq} \left( \sqrt{xy} \cdot \sqrt{\frac{x}{y} + 1} \cdot 1 \right)^2 = (x+1)^2 \Leftrightarrow$$

$$(xy + 1) \left( \frac{x}{y} + 1 \right) \geq (x+1)^2 \Leftrightarrow \frac{1}{(x+1)^2} \geq \frac{1}{(xy+1) \left( \frac{x}{y} + 1 \right)} \Leftrightarrow$$

$$\frac{1}{(x+1)^2} \geq \frac{y}{(1+xy)(x+y)}$$



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*Similarly:*

$$\frac{1}{(y+1)^2} \geq \frac{x}{(1+xy)(x+y)} \Rightarrow \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} \geq \frac{x+y}{(1+xy)(x+y)} = \frac{1}{1+xy}; \quad (1)$$

*Similarly:*

$$\frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} \geq \frac{1}{1+yz}; \quad (2)$$

$$\frac{1}{(x+1)^2} + \frac{1}{(z+1)^2} \geq \frac{1}{1+zx}; \quad (3)$$

*From (1),(2),(3) we have:*

$$2 \left( \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} \right) \geq \frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \Leftrightarrow \\ \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} \geq \frac{1}{2} \left( \frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \right)$$

*On the other hand, we have:*

$$\frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \stackrel{\text{Bergstrom}}{\geq} \frac{(1+1+1)^2}{3+xy+yz+zx} = \frac{9}{3+xy+yz+zx}$$

*Therefore,*

$$\begin{aligned} & \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} + \frac{xy+yz+zx}{8} \geq \\ & \geq \frac{9}{2(3+xy+yz+zx)} + \frac{xy+yz+zx}{8} = \\ & = \frac{9}{2(3+xy+yz+zx)} + \frac{xy+yz+zx+3}{8} - \frac{3}{8} \stackrel{\text{AM-GM}}{\geq} \\ & \stackrel{\text{AM-GM}}{\geq} 2 \cdot \sqrt{\frac{9}{2(3+xy+yz+zx)} \cdot \frac{xy+yz+zx+3}{8}} - \frac{3}{8} = 2 \sqrt{\frac{9}{16}} - \frac{3}{8} = \frac{9}{8} \end{aligned}$$

**Solution 2 by Abdul Aziz-Semarang-Indonesia**

$$(xy-1)^2 + xy(x-y)^2 \geq 0 \Leftrightarrow \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} \geq \frac{1}{1+xy}$$

*Similarly:*

$$\frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} \geq \frac{1}{1+yz}$$



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$$\frac{1}{(x+1)^2} + \frac{1}{(z+1)^2} \geq \frac{1}{1+zx}$$

*Applying AM-GM inequality, we get:*

$$\frac{1}{2} \left( \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} \right) + \frac{1+xy}{8} \geq \frac{1}{2(1+xy)} + \frac{1+xy}{8} \stackrel{\text{AM-GM}}{\geq} \frac{1}{2}$$

$$\frac{1}{2} \left( \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} \right) + \frac{1+yz}{8} \geq \frac{1}{2(1+yz)} + \frac{1+yz}{8} \stackrel{\text{AM-GM}}{\geq} \frac{1}{2}$$

$$\frac{1}{2} \left( \frac{1}{(z+1)^2} + \frac{1}{(x+1)^2} \right) + \frac{1+zx}{8} \geq \frac{1}{2(1+zx)} + \frac{1+zx}{8} \stackrel{\text{AM-GM}}{\geq} \frac{1}{2}$$

*Adding up three inequalities, we get:*

$$\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} + \frac{xy+yz+zx}{8} + \frac{3}{8} \geq \frac{3}{2}$$

$$\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} + \frac{xy+yz+zx}{8} \geq \frac{9}{8}$$

*Equality holds when  $x = y = z = 1$ .*

**662. If  $x, y, z > 0, xy + yz + zx = 3$  then:**

$$13(x^2 + y^2 + z^2) + 27x^2y^2z^2 \geq 66$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

$$x, y, z > 0, xy + yz + zx = 3 \Leftrightarrow \frac{x}{\sqrt{3}} \cdot \frac{y}{\sqrt{3}} + \frac{y}{\sqrt{3}} \cdot \frac{z}{\sqrt{3}} + \frac{z}{\sqrt{3}} \cdot \frac{x}{\sqrt{3}} = 1 \Rightarrow$$

$$\exists \Delta ABC \text{ such that: } \frac{x}{\sqrt{3}} = \tan \frac{A}{2}; \frac{y}{\sqrt{3}} = \tan \frac{B}{2}; \frac{z}{\sqrt{3}} = \tan \frac{C}{2} \Rightarrow$$

$$x = \sqrt{3} \tan \frac{A}{2}; y = \sqrt{3} \tan \frac{B}{2}; z = \sqrt{3} \tan \frac{C}{2}$$

*Hence,*

$$13(x^2 + y^2 + z^2) + 27x^2y^2z^2 \geq 66 \Leftrightarrow$$

$$13 \cdot 3 \sum_{cyc} \tan^2 \frac{A}{2} + 27 \cdot 27 \cdot \left( \prod_{cyc} \tan \frac{A}{2} \right)^2 \geq 66 \Leftrightarrow$$



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$$13 \sum_{cyc} \tan^2 \frac{A}{2} + 27 \cdot 9 \cdot \left( \prod_{cyc} \tan \frac{A}{2} \right)^2 \geq 22 \Leftrightarrow$$

$$13 \cdot \frac{(4R+r)^2 - 2s^2}{s^2} + 27 \cdot 9 \left( \frac{r}{s} \right)^2 \geq 22 \Leftrightarrow 13(4R+r)^2 + 243r^2 \stackrel{(*)}{\geq} 48s^2$$

*But:  $s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}$*

*Let:  $t = \frac{R}{r} \geq 2$ , we must show that:*

$$13(4t+1)^2 + 243 \geq 48 \left[ (2t^2 + 10t - 1) + (2t-4)\sqrt{t^2 - 2t} \right] \Leftrightarrow$$

$$13(16t^2 + 8t + 1) + 243 - (96t^2 + 480t - 48) \geq 48(2t-4)\sqrt{t^2 - 2t} \Leftrightarrow$$

$$112t^2 - 376t + 304 \geq 48 \cdot 2(t-2)\sqrt{t^2 - 2t} \Leftrightarrow$$

$$(t-2)(14t-9) \geq 12(t-2)\sqrt{t^2 - 2t}$$

*Because:  $t \geq 2 \Rightarrow t-2 \geq 0$ . We need to prove:*

$$14t - 9 > 12\sqrt{t^2 - 2t} \Leftrightarrow (14t - 9)^2 > 12^2(t^2 - 9) \Leftrightarrow$$

$$52t^2 + 36t + 81 > 0 \text{ true for } t \geq 2 \Rightarrow (*) \text{ is true. Proved.}$$

**663. If  $x, y, z > 0$ ,  $x^3 + y^3 + z^3 = 24$  then:**

$$(x^5y + y^5z + z^5x)(x^2y + y^2z + z^2x) \geq 576xyz$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

*We have:  $576 = 24^2 = (x^3 + y^3 + z^3)^2$*

*Inequality becomes:*

$$(x^5y + y^5z + z^5x)(x^2y + y^2z + z^2x) \geq (x^3 + y^3 + z^3)^2xyz; \quad (1)$$

$$\begin{aligned} x^5y + y^5z + z^5x &= \frac{x^6}{y} + \frac{y^6}{z} + \frac{z^6}{x} = \frac{(x^3)^2}{y} + \frac{(y^3)^2}{z} + \frac{(z^3)^2}{x} \stackrel{\text{Bergstrom}}{\geq} \\ &\geq \frac{(x^3 + y^3 + z^3)^2}{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}} = \frac{(x^3 + y^3 + z^3)^2xyx}{x^2z + xy^2 + yz^2} \end{aligned}$$

*WLOG, suppose:  $x \geq y \geq z$ . We need to prove:*

$$x^2y + y^2z + z^2x \geq x^2z + xy^2 + yz^2 \Leftrightarrow$$



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$(x - y)(x - z)(y - z) \geq 0$  true by  $x \geq y \geq z \Rightarrow x - y \geq 0; x - z \geq 0; y - z \geq 0$

$\Rightarrow (1)$  is true. Proved.

**664. If  $x, y, z > 0$  prove that:**

$$\frac{(x+y)(y+z)(z+x)}{8xyz} \geq \frac{1}{2} + \frac{1}{3} \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)$$

*Proposed by Alex Szoros-Romania*

**Solution 1 by Marian Dincă-Romania**

$$\begin{aligned} \frac{(x+y)(y+z)(z+x)}{8xyz} &= \frac{(x+y+z)(xy+yz+zx) - xyz}{8xyz} = \\ &= \frac{1}{8}(x+y+z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) - \frac{1}{8} \\ \frac{1}{3} \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) &= \frac{1}{3} \left[ \left( \frac{x}{y+z} + 1 \right) + \left( \frac{y}{z+x} + 1 \right) + \left( \frac{z}{x+y} + 1 \right) - 3 \right] = \\ &= \frac{1}{3}(x+y+z) \left( \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) - 1 \end{aligned}$$

So, we have:

$$\begin{aligned} \frac{1}{8}(x+y+z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) - \frac{1}{8} &\geq \frac{1}{2} + \frac{1}{3}(x+y+z) \left( \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) - 1 \Leftrightarrow \\ \frac{1}{8}(x+y+z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) + \frac{3}{8} &\geq \frac{1}{3}(x+y+z) \left( \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) \Leftrightarrow \\ (x+y+z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) + 3 &\geq \frac{8}{3}(x+y+z) \left( \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) \Leftrightarrow \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{3}{x+y+z} &\geq \frac{8}{3} \left( \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right); \quad (1) \end{aligned}$$

And use the well-known inequality:

$$f(x) + f(y) + f(z) + f\left(\frac{x+y+z}{3}\right) \geq \frac{4}{3} \left( f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right); \quad (*)$$

for any convex function.

Let  $f(t) = \frac{1}{t}, t > 0$ , following inequality (1)

Consequence of Tiberiu Popoviciu's inequality



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$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 2\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right)$$

$$\text{And } f(x) + f(y) + f(z) \geq f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)$$

$$\begin{aligned} \text{Then } f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) &\geq 2(f(x) + f(y) + f(z)) \geq \\ &\geq 4\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right) \text{ and obtain (*).} \end{aligned}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\text{Let } a = x + y; b = x + z; c = x + y \Rightarrow x = \frac{b+c-a}{2} > 0; y = \frac{a+c-b}{2} > 0; z = \frac{a+b-c}{2} > 0 \Rightarrow$$

$$b + c > a; a + c - b > 0; a + b > c \Rightarrow \exists \Delta ABC \text{ such that } AB = c; BC = a; CA = b.$$

*Inequality becomes as:*

$$\begin{aligned} \frac{abc}{8\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)} &\geq \frac{1}{2} + \frac{1}{3}\left(\frac{b+c-a}{2a} + \frac{a+c-b}{2b} + \frac{a+b-c}{2c}\right) \Leftrightarrow \\ \frac{4Rrs}{8sr^2} &\geq \frac{1}{2} + \frac{1}{3}\left(\frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c}\right) \Leftrightarrow \\ \frac{R}{2r} &\geq \frac{1}{2} + \frac{1}{3} \cdot \frac{s^2 - 8Rr + r^2}{4Rr}; \left( \because \sum_{cyc} \frac{s-a}{a} = \frac{s^2 - 8Rr + r^2}{4Rr} \right) \end{aligned}$$

$$6R^2 \geq 6Rr + s^2 - 8Rr + r^2 \Leftrightarrow 6R^2 + 2Rr - r^2 \geq s^2; \quad (1)$$

$$\text{But: } s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen)}$$

*We need to prove:*

$$4R^2 + 4Rr + 3r^2 \leq 6R^2 + 2Rr - r^2 \Leftrightarrow R^2 - Rr - 2r^2 \geq 0 \Leftrightarrow$$

$$(R - 2r)(R + r) \geq 0 \text{ true by } R \geq 2r \text{ (Euler)} \Rightarrow (1) \text{ true. Proved.}$$

**Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $x, y, z > 0$ , we have:*

$$\begin{aligned} \frac{(x+y)(y+z)(z+x)}{8xyz} &\geq \frac{1}{2} + \frac{1}{3}\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right) \Leftrightarrow \\ 3\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{z} + \frac{z}{y} + \frac{y}{x} + 2\right) &\geq 12 + 8\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right) \Leftrightarrow \\ 3\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{z} + \frac{z}{y} + \frac{y}{x}\right) &\geq 6 + 8\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right) \Leftrightarrow \end{aligned}$$



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$$3 \left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{z} + \frac{z}{y} + \frac{y}{x} \right) \geq 6 + 8 \left( \frac{x}{4y} + \frac{x}{4z} + \frac{y}{4x} + \frac{y}{4z} + \frac{z}{4x} + \frac{z}{4y} \right) \Leftrightarrow$$

$$3 \left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{z} + \frac{z}{y} + \frac{y}{x} \right) \geq 6 + 2 \left( \frac{x}{y} + \frac{x}{z} + \frac{y}{x} + \frac{y}{z} + \frac{z}{x} + \frac{z}{y} \right) \Leftrightarrow$$

$$\frac{x}{y} + \frac{y}{x} + \frac{z}{x} + \frac{x}{z} + \frac{y}{z} + \frac{z}{y} \geq 6, \forall x, y, z > 0 \text{ (true).}$$

**665. If  $a, b, c > 0$  such that  $abc = 1$  and  $n \geq 0$  then prove:**

$$\frac{1}{na^2 + a} + \frac{1}{nb^2 + b} + \frac{1}{nc^2 + c} \geq \frac{4}{n+1} \left( \frac{1}{a^2 + b^2 + 2} + \frac{1}{b^2 + c^2 + 2} + \frac{1}{c^2 + a^2 + 2} \right)$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

*Because:  $a, b, c > 0, abc = 1$  we have:*

$$\sum_{\text{cyc}} \frac{1}{na^2 + a} = \sum_{\text{cyc}} \frac{\left(\frac{1}{a}\right)^2}{n + \frac{1}{a}} = \sum_{\text{cyc}} \frac{(bc)^2}{n + bc} \stackrel{\text{Bergstrom}}{\geq} \frac{(ab + bc + ca)^2}{3n + (ab + bc + ca)}; (1)$$

$$a^2 + b^2 + 2 = (a^2 + 1) + (b^2 + 1) \stackrel{\text{AM-GM}}{\geq} 2a + 2b = 2(a + b); (\text{and analogs}) \Rightarrow$$

$$\sum_{\text{cyc}} \frac{1}{a^2 + b^2 + 2} \leq \frac{1}{2} \sum_{\text{cyc}} \frac{1}{a + b} \stackrel{\text{Bergstrom}}{\leq} \frac{1}{2} \cdot \frac{1}{4} \sum_{\text{cyc}} \left( \frac{1}{a} + \frac{1}{b} \right) = \frac{1}{4} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) =$$

$$= \frac{1}{4} \cdot \frac{ab + bc + ca}{abc} \stackrel{abc=1}{=} \frac{ab + bc + ca}{4} \Rightarrow$$

$$\frac{4}{n+1} \sum_{\text{cyc}} \frac{1}{a^2 + b^2 + 2} \leq \frac{ab + bc + ca}{n+1}; (2)$$

$$\text{Let } t = ab + bc + ca \stackrel{\text{AM-GM}}{\geq} 3\sqrt[3]{(abc)^2} = 3.$$

*From (1),(2) we need to prove:*

$$\frac{t^2}{3n+t} \geq \frac{t}{n+1} \Leftrightarrow (n+1)t^2 \geq 3nt + t^2 \Leftrightarrow nt^2 \geq 3nt \Leftrightarrow$$

$$nt(t-3) \geq 0 \text{ true by } t \geq 3, n \geq 0. \text{ Proved.}$$

**666. If  $a, b, c, d > 0$  such that  $abcd = 16$  and  $n \in \mathbb{N}$  then prove that:**

$$a^n \sqrt{b+c} + b^n \sqrt{c+d} + c^n \sqrt{d+a} + d^n \sqrt{a+b} \leq \frac{a^{2n+1} + b^{2n+1} + c^{2n+1} + d^{2n+1}}{2^n}$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**



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$$\begin{aligned}
 a^n\sqrt{b+c} + b^n\sqrt{c+a} + c^n\sqrt{d+a} + d^n\sqrt{a+b} &\stackrel{BCS}{\leq} \sqrt{\sum_{cyc}(a^n)^2 \cdot \sum_{cyc}(a+b)} = \\
 &= \sqrt{2 \sum_{cyc} a^{2n} \cdot \sum_{cyc} a} \\
 \frac{a^{2n+1} + b^{2n+1} + c^{2n+1} + d^{2n+1}}{2^n} &= \frac{\sum_{cyc}(a^{2n} \cdot a)}{2^n} \stackrel{Chebyshev's}{\geq} \frac{\frac{1}{4} \cdot \sum_{cyc} a^{2n} \cdot \sum_{cyc} a}{2^n} \Leftrightarrow \\
 \frac{a^{2n+1} + b^{2n+1} + c^{2n+1} + d^{2n+1}}{2^n} &\stackrel{Chebyshev's}{\geq} \frac{\sum_{cyc} a^{2n} \cdot \sum_{cyc} a}{2^{n+2}} \\
 \text{Let } t = \sqrt{2 \sum_{cyc} a^{2n} \cdot \sum_{cyc} a} &\stackrel{AM-GM}{\geq} \sqrt{2 \cdot 4 \cdot \sqrt[4]{(abcd)^{2n}} \cdot 4\sqrt[4]{abcd}} = \\
 &= \sqrt{2 \cdot 4\sqrt[4]{16^{2n}} \cdot 4\sqrt[4]{16}} = \sqrt{2 \cdot 4 \cdot 2^{2n} \cdot 4 \cdot 2} = 2 \cdot 4 \cdot 2^n = 2^{n+3}
 \end{aligned}$$

We need to prove:  $t \leq \frac{t^2}{2 \cdot 2^{n+2}} \Leftrightarrow t^2 - 2^{n+3}t \geq 0 \Leftrightarrow t(t - 2^{n+3}) \geq 0$  true by  $t \geq 2^{n+3}$ .

**667. If  $a, b, c > 0$  such that  $abc = 8$  and  $n \in \mathbb{N}$  then prove:**

$$a^n\sqrt{b+c} + b^n\sqrt{c+a} + c^n\sqrt{a+b} \leq \frac{a^{2n+1} + b^{2n+1} + c^{2n+1}}{2^n}$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

For  $n \in \mathbb{N}, a, b, c > 0$  we have:

$$\begin{aligned}
 a^n\sqrt{b+c} + b^n\sqrt{c+a} + c^n\sqrt{a+b} &\stackrel{BCS}{\leq} \sqrt{(a^n)^2 + (b^n)^2 + (c^n)^2} \cdot \sqrt{2(a+b+c)} = \\
 &= \sqrt{2(a+b+c)(a^{2n} + b^{2n} + c^{2n})}; \quad (1) \\
 \frac{a^{2n+1} + b^{2n+1} + c^{2n+1}}{2^n} &= \frac{a^{2n} \cdot a + b^{2n} \cdot b + c^{2n} \cdot c}{2^n} \stackrel{Chebyshev's}{\geq} \\
 &> \frac{\frac{1}{3}(a^{2n} + b^{2n} + c^{2n})(a+b+c)}{2^n} = \frac{(a^{2n} + b^{2n} + c^{2n})(a+b+c)}{3 \cdot 2^n}; \quad (2)
 \end{aligned}$$

$$\text{Let } t = \sqrt{2(a+b+c)(a^{2n} + b^{2n} + c^{2n})} \stackrel{AM-GM}{\geq} \sqrt{2 \cdot 3\sqrt[3]{abc} \cdot 3\sqrt[3]{(abc)^{2n}}} = 3 \cdot 2^{n+1}$$



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*From (1),(2) we need to prove:*

$$t \leq \frac{t^2}{2 \cdot 3 \cdot 2^n} \Leftrightarrow t^2 \geq 2 \cdot 3 \cdot 2^n t \Leftrightarrow t(t - 3 \cdot 2^{n+1}) \geq 0 \text{ true for } t \geq 3 \cdot 2^{n+1}.$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $a, b, c > 0, abc = 8, n \in \mathbb{N}$  we have:*

$$\begin{aligned} a^n\sqrt{b+c} + b^n\sqrt{c+a} + c^n\sqrt{a+b} &= \sqrt{a^{2n}(b+c)} + \sqrt{b^{2n}(c+a)} + \sqrt{c^{2n}(a+b)} \leq \\ &\leq \sqrt{3(a^{2n}(b+c) + b^{2n}(c+a) + c^{2n}(a+b))} \leq \sqrt{6(a^{2n+1} + b^{2n+1} + c^{2n+1})} \leq \\ &\leq \frac{a^{2n+1} + b^{2n+1} + c^{2n+1}}{2^n}. \text{ Iff} \\ 6(a^{2n+1} + b^{2n+1} + c^{2n+1}) &\leq \frac{(a^{2n+1} + b^{2n+1} + c^{2n+1})^2}{2^{2n}} \Leftrightarrow \\ 6 \cdot 2^{2n} &\leq a^{2n+1} + b^{2n+1} + c^{2n+1} \Leftrightarrow 6 \cdot 2^{2n} \leq \frac{(a+b+c)^{2n+1}}{3^{2n+1}-1} \Leftrightarrow \\ 6 \cdot 2^n \cdot 3^n &\leq (a+b+c)^{2n+1} \Leftrightarrow 2^{n+1} \cdot 3^{n+1} \leq (a+b+c)^{2n+1} \text{ true from} \\ abc = 8 &\Rightarrow a+b+c \geq 6. \end{aligned}$$

**668. If  $a, b, c > 0, a^2 + b^2 + c^2 = 3, \mu \geq 1$  then:**

$$\frac{1}{\mu + a^2} + \frac{1}{\mu + b^2} + \frac{1}{\mu + c^2} \leq \frac{27}{(1+\mu)(a+b+c)^2}$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

*For  $\mu \geq 1$  we have:*

$$\begin{aligned} \frac{1}{\mu + a^2} + \frac{1}{\mu + b^2} + \frac{1}{\mu + c^2} &\leq \frac{27}{(1+\mu)(a+b+c)^2} \\ \left(\frac{1}{\mu + a^2} - \frac{1}{\mu}\right) + \left(\frac{1}{\mu + b^2} - \frac{1}{\mu}\right) + \left(\frac{1}{\mu + c^2} - \frac{1}{\mu}\right) + \frac{3}{\mu} &\leq \frac{27}{(1+\mu)(a+b+c)^2} \\ \frac{a^2}{\mu + a^2} + \frac{b^2}{\mu + b^2} + \frac{c^2}{\mu + c^2} + \frac{27\mu}{(1+\mu)(a+b+c)^2} &\geq 3; \quad (1) \\ \frac{a^2}{\mu + a^2} + \frac{b^2}{\mu + b^2} + \frac{c^2}{\mu + c^2} &\stackrel{\text{Bergstrom}}{\geq} \frac{(a+b+c)^2}{3\mu + a^2 + b^2 + c^2} \stackrel{a^2+b^2+c^2=3}{=} \end{aligned}$$



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$$= \frac{(a+b+c)^2}{\mu(a^2+b^2+c^2)+a^2+b^2+c^2} = \frac{(a+b+c)^2}{(\mu+1)(a^2+b^2+c^2)}; \quad (2)$$

$$\frac{27\mu}{(1+\mu)(a+b+c)^2} = \frac{9 \cdot 3\mu}{(1+\mu)(a+b+c)^2} \stackrel{a^2+b^2+c^2=3}{=} \frac{9\mu(a^2+b^2+c^2)}{(1+\mu)(a+b+c)^2}; \quad (3)$$

*Let  $x = a^2 + b^2 + c^2$ ;  $y = (a+b+c)^2$ ;  $(3x \geq y > 0)$*

*From (2),(3) we need to prove:*

$$\frac{y}{(\mu+1)x} + \frac{9\mu x}{(\mu+1)y} \geq 3 \Leftrightarrow y^2 + 9\mu x^2 \geq 3(\mu+1)xy \Leftrightarrow$$

$$y^2 + 9x^2 + (9\mu - 9)x^2 \geq 6xy + (3\mu - 3)xy \Leftrightarrow$$

$$(y - 3x)^2 + 3(\mu - 1)(3x^2 - xy) \geq 0 \Leftrightarrow$$

*$(y - 3x)^2 + 3x(\mu - 1)(3x - y) \geq 0$  which is true because:*

$$\mu \geq 1 \Rightarrow \mu - 1 \geq 0; 3x \geq y \Rightarrow 3x - y \geq 0 \Rightarrow 3x(\mu - 1)(3x - y) \geq 0;$$

*$(y - 3x)^2 \geq 0 \Rightarrow (1) \text{ is true. Proved.}$*

**669. If  $a, b, c > 0; m, n \in \mathbb{N}^*$  then prove:**

$$(i) n > m: \frac{a^n + b^n}{a^m + b^m} + \frac{b^n + c^n}{b^m + c^m} + \frac{c^n + a^n}{c^m + a^m} \geq (ab)^{\frac{n-m}{2}} + (bc)^{\frac{n-m}{2}} + (ca)^{\frac{n-m}{2}}$$

$$(ii) n < m: \frac{a^n + b^n}{a^m + b^m} + \frac{b^n + c^n}{b^m + c^m} + \frac{c^n + a^n}{c^m + a^m} \leq (ab)^{\frac{n-m}{2}} + (bc)^{\frac{n-m}{2}} + (ca)^{\frac{n-m}{2}}$$

*Proposed by Pavlos Trifon-Greece*

**Solution by proposer**

$$(i) (n, 0), \left( \frac{n+m}{2}, \frac{n-m}{2} \right); (\text{by Murihead}) \Rightarrow$$

$$a^n b^0 + a^0 b^n \geq a^{\frac{n+m}{2}} b^{\frac{n-m}{2}} + a^{\frac{n-m}{2}} b^{\frac{n+m}{2}}$$

$$a^n + b^n \geq (ab)^{\frac{n-m}{2}} \cdot (a^m + b^m) \Rightarrow$$

$$\frac{a^n + b^n}{a^m + b^m} \geq (ab)^{\frac{n-m}{2}}; \frac{b^n + c^n}{b^m + c^m} \geq (bc)^{\frac{n-m}{2}}; \frac{c^n + a^n}{c^m + a^m} \geq (ca)^{\frac{n-m}{2}}$$

$$\frac{a^n + b^n}{a^m + b^m} + \frac{b^n + c^n}{b^m + c^m} + \frac{c^n + a^n}{c^m + a^m} \geq (ab)^{\frac{n-m}{2}} + (bc)^{\frac{n-m}{2}} + (ca)^{\frac{n-m}{2}}$$

$$(ii) (m, 0), \left( \frac{n+m}{2}, \frac{m-n}{2} \right); (\text{by Murihead}) \Rightarrow$$



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$$a^m b^0 + a^0 b^m \geq a^{\frac{n+m}{2}} b^{\frac{m-n}{2}} + a^{\frac{m-n}{2}} b^{\frac{n+m}{2}}$$

$$a^m + b^m \geq (ab)^{\frac{m-n}{2}} \cdot (a^n + b^n) \Rightarrow$$

$$\frac{a^m + b^m}{a^n + b^n} \geq (ab)^{\frac{m-n}{2}}; \frac{b^m + c^m}{b^n + c^n} \geq (bc)^{\frac{m-n}{2}}; \frac{c^m + a^m}{c^n + a^n} \geq (ca)^{\frac{m-n}{2}}$$

$$\frac{a^n + b^n}{a^m + b^m} + \frac{b^n + c^n}{b^m + c^m} + \frac{c^n + a^n}{c^m + a^m} \leq (ab)^{\frac{n-m}{2}} + (bc)^{\frac{n-m}{2}} + (ca)^{\frac{n-m}{2}}$$

**670. If  $a, b, c > 0, abc(a + b + c)^3 = 27$  then:**

$$(a + b)(b + c)(c + a) \geq 8$$

*Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

$$27 = abc(a + b + c)^3 \stackrel{AM-GM}{\leq} \frac{(a + b + c)^3}{27} \cdot (a + b + c)^3 = \frac{(a + b + c)^6}{27} \Rightarrow (a + b + c)^6 \geq 27^2 \Rightarrow a + b + c \geq 3$$

$$(ab + bc + ca)^2 \geq 3abc(a + b + c) = 3 \cdot \frac{27}{(a + b + c)^3} \cdot (a + b + c) = \frac{3^4}{(a + b + c)^2} \\ \Rightarrow ab + bc + ca \geq \frac{9}{ab + bc + ca}$$

$$\text{Now, } (a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc = \\ = (a + b + c)(ab + bc + ca) - \frac{27}{(a + b + c)^3} \geq \frac{9(a + b + c)}{a + b + c} - \frac{27}{(a + b + c)^3} = \\ = 9 - \frac{27}{(a + b + c)^3} \stackrel{a+b+c \geq 3}{\geq} 9 - \frac{27}{3^3} = 8$$

**Solution 2 by Tri Vuduc-Vietnam**

$$(a + b)(b + c)(c + a) \geq 8 \Leftrightarrow 9(a + b)(b + c)(c + a) \geq 72$$

$$\text{Use } 9(a + b)(b + c)(c + a) \geq 8(a + b + c)(ab + bc + ca) \Rightarrow$$

$$(a + b + c)(ab + bc + ca) \geq 9; (*) \text{ is true, because:}$$

$$abc(a + b + c)^3 = 27 \Rightarrow 27abc(a + b + c)^3 = 27^2 \Rightarrow$$

$$(a + b + c)^6 \geq 27^2 \Rightarrow a + b + c \geq 3$$



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$$\begin{aligned}
 (ab + bc + ca)^2 &\geq 3abc(a + b + c) = \frac{3abc(a + b + c)^3}{(a + b + c)^3} \cdot (a + b + c) = \\
 &= 3 \cdot \frac{27}{(a + b + c)^2} \Rightarrow (*) \text{true}.
 \end{aligned}$$

**671. If  $x, y, z > 0, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3, \mu \geq 0$  then:**

$$(\mu x + 1)(\mu y + 1)(\mu z + 1) \leq (\mu x + y)(\mu y + z)(\mu z + x)$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Marian Ursărescu-Romania**

Let:  $\frac{1}{x} = a; \frac{1}{y} = b; \frac{1}{z} = c \Rightarrow a + b + c = 3$ .

**We must show that:**

$$\begin{aligned}
 \left(\frac{\mu}{a} + 1\right)\left(\frac{\mu}{b} + 1\right)\left(\frac{\mu}{c} + 1\right) &\leq \left(\frac{\mu}{a} + \frac{1}{b}\right)\left(\frac{\mu}{b} + \frac{1}{c}\right)\left(\frac{\mu}{c} + \frac{1}{a}\right) \Leftrightarrow \\
 abc(\mu + a)(\mu + b)(\mu + c) &\leq (\mu b + a)(\mu c + b)(\mu a + c); (1)
 \end{aligned}$$

**From Huygens inequality, we have:**

$$\begin{aligned}
 (\mu + a)(\mu + b)(\mu + c) &\geq \left(\sqrt[3]{\mu^3 abc} + \sqrt[3]{abc}\right)^3 \Leftrightarrow \\
 (\mu + a)(\mu + b)(\mu + c) &\geq abc(1 + \mu)^3; (2)
 \end{aligned}$$

**From (1),(2) we must show that:**

$$(\mu + a)(\mu + b)(\mu + c) \leq (1 + \mu)^3 \Leftrightarrow \sqrt[3]{(\mu + a)(\mu + b)(\mu + c)} \leq 1 + \mu; (3)$$

**But:**

$$\sqrt[3]{(\mu + a)(\mu + b)(\mu + c)} \leq \frac{3\mu + a + b + c}{3} = \frac{3\mu + 3}{3} = \mu + 1 \Rightarrow (3) \text{is true.}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $x, y, z > 0, \mu \geq 0$  and  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3 \Rightarrow xy + yz + zx = 3xyz$

$$1) x^2y + y^2z + z^2x \geq 3xyz = xy + yz + zx \text{ iff } \frac{x}{z} + \frac{y}{x} + \frac{z}{y} \geq 3 \text{ (true)}$$

$$2) xyz \geq 1; \left(3 \geq \sqrt[3]{xyz}\right)$$



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$$3) x^2z + z^2y + y^2x \geq \frac{(xz+zy+yx)(x+y+z)}{3} = \frac{3xyz(x+y+z)}{3} = \\ = xyz(x+y+z) \geq x+y+z; (xyz \geq 1)$$

$$(\mu x + 1)(\mu y + 1)(\mu z + 1) = \mu^3 xyz + \mu^2(xy + yz + zx) + \mu(x + y + z) + 1 \\ (\mu x + y)(\mu y + z)(\mu z + x) = \\ = \mu^3 xyz + \mu^2(x^2y + y^2z + z^2x) + \mu(x^2z + z^2y + y^2x) + xyz$$

*Hence*

$$(\mu x + 1)(\mu y + 1)(\mu z + 1) \leq (\mu x + y)(\mu y + z)(\mu z + x)$$

**672. If  $a, b, c > 0, a + b + c = 3$  then:**

$$\frac{1}{9}(\sqrt{a} + \sqrt{b} + \sqrt{c}) + \sum_{cyc} \frac{a}{7bc + \sqrt{2(b^4 + c^4)}} \geq \frac{2}{3}$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

*For  $a, b > 0$  we have:*

$$2(a^4 + b^4) \leq (3a^2 - 4ab + 3b^2)^2 \Leftrightarrow (a - b)^2(7a^2 - 10ab + 7b^2) \geq 0 \Leftrightarrow \\ (a - b)^4[7(a - b)^2 + 4ab] \geq 0 \text{ true for } a, b > 0$$

*Equality if  $a = b$ . Similarly:*

$$2(b^4 + c^4) \leq (3b^2 - 4bc + 3c^2)^2$$

$$2(c^4 + a^4) \leq (3c^2 - 4ca + 3a^2)^2$$

$$\Omega_1 = \sum_{cyc} \frac{a}{7bc + \sqrt{2(b^4 + c^4)}} \geq \sum_{cyc} \frac{a}{3b^2 - 4bc + 3c^2 + 7bc} = \sum_{cyc} \frac{a}{3(b^2 + bc + c^2)} = \Omega_2$$

$$\Omega_3 = \sum_{cyc} \frac{a}{b^2 + bc + c^2} = \sum_{cyc} \frac{a^2}{ab^2 + abc + ac^2} \stackrel{\text{Bergstrom}}{\geq}$$

$$\geq \frac{(a + b + c)^2}{ab(a + b) + bc(b + c) + ca(c + a) + 3abc} = \frac{(a + b + c)^2}{(a + b + c)(ab + bc + ca)} = \\ = \frac{3}{ab + bc + ca}. \text{ Therefore,}$$

$$\Omega_1 \geq \Omega_2 \geq \Omega_3 = \frac{1}{3} \cdot \frac{3}{ab + bc + ca} = \frac{1}{ab + bc + ca}$$

*Other, using AM-GM we have:*



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$$\sqrt{a} + \sqrt{a} + a^2 \geq 3\sqrt[3]{(\sqrt{a})^2 a^2} = 3a \Rightarrow 2\sqrt{a} + a^2 \geq 3a; (1)$$

*Similarly:*

$$2\sqrt{b} + b^2 \geq 3b; (2)$$

$$2\sqrt{c} + c^2 \geq 3c; (3)$$

*From (1),(2),(3) we have:*

$$2(\sqrt{a} + \sqrt{b} + \sqrt{c}) + a^2 + b^2 + c^2 \geq 3(a + b + c) \Leftrightarrow$$

$$2(\sqrt{a} + \sqrt{b} + \sqrt{c}) + (a + b + c)^2 - 2(ab + bc + ca) \geq 3(a + b + c) \Leftrightarrow$$

$$2(\sqrt{a} + \sqrt{b} + \sqrt{c}) + 3^2 - 2(ab + bc + ca) \geq 3; (\because a + b + c = 3) \Leftrightarrow$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

*So,*

$$\begin{aligned} LHS &= \frac{1}{9}(\sqrt{a} + \sqrt{b} + \sqrt{c}) + \Omega_1 \geq \frac{1}{9}(ab + bc + ca) + \frac{1}{ab + bc + ca} \stackrel{AM-GM}{\geq} \\ &\geq 2\sqrt{\frac{1}{9}(ab + bc + ca) \cdot \frac{1}{ab + bc + ca}} = \frac{2}{3} \end{aligned}$$

*Equality holds if and only if  $a = b = c = 1$ .*

**673. If  $a_i > 0, i \in \overline{1, n}, n \in \mathbb{N}, n \geq 2$  then:**

$$(n-1)^{2(n-1)} \left( \prod_{i=1}^n a_i \right)^2 \left( \sum_{i=1}^n \frac{a_i}{\left( \sum_{\substack{j=1 \\ j \neq i}}^n a_j \right)^{\frac{n-1}{2}}} \right)^4 \leq n \left( \sum_{i=1}^n a_i^2 \right)^3$$

*Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan*

**Solution by Tran Hong-Dong Thap-Vietnam**

*With  $a_i > 0, (i = 1, 2, \dots, n, n \in \mathbb{N}, n \geq 2)$ . By AM- GM we have:*

- $\sum_{j=1, j \neq i}^n a_j \geq (n-1) \cdot \sqrt[n-1]{\prod_{j=1, j \neq i}^n a_j}$



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$$\begin{aligned}
 & \rightarrow \left( \sum_{j=1, i \neq j}^n a_j \right)^{n-1} \geq (n-1)^{n-1} \cdot \prod_{j=1, j \neq i}^n a_j \\
 & \rightarrow \left( \sum_{j=1, i \neq j}^n a_j \right)^{\frac{n-1}{2}} \geq (n-1)^{\frac{n-1}{2}} \cdot \sqrt[n]{\prod_{j=1, j \neq i}^n a_j} \\
 & \rightarrow \sum_{i=1}^n \frac{a_i}{(\sum_{j=1, i \neq j}^n a_j)^{\frac{n-1}{2}}} \leq \sum_{i=1}^n \frac{a_i}{(n-1)^{\frac{n-1}{2}} \cdot \sqrt{\prod_{j=1, j \neq i}^n a_j}} \\
 & = \frac{1}{(n-1)^{\frac{n-1}{2}}} \cdot \sum_{i=1}^n \frac{a_i}{\sqrt{\prod_{j=1, j \neq i}^n a_j}} \stackrel{BCS}{\leq} \frac{1}{(n-1)^{\frac{n-1}{2}}} \cdot \left( \sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n \frac{1}{\prod_{j=1, j \neq i}^n a_j}} \right) \\
 & = \frac{1}{(n-1)^{\frac{n-1}{2}}} \cdot \sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\frac{\sum_{i=1}^n a_i}{\prod_{i=1}^n a_i}} = \frac{1}{(n-1)^{\frac{n-1}{2}}} \cdot \sqrt{\sum_{i=1}^n a_i^2} \cdot \frac{\sqrt{\sum_{i=1}^n a_i}}{\sqrt{\prod_{i=1}^n a_i}} \\
 & \stackrel{BCS}{\leq} \frac{1}{(n-1)^{\frac{n-1}{2}}} \cdot \sqrt{\sum_{i=1}^n a_i^2} \cdot \frac{\sqrt{n(\sum_{i=1}^n a_i^2)}}{\sqrt{\prod_{i=1}^n a_i}} \\
 & (n-1)^{2(n-1)} \left( \prod_{i=1}^n a_i \right)^2 \cdot \left( \sum_{i=1}^n \frac{a_i}{(\sum_{j=1, i \neq j}^n a_j)^{\frac{n-1}{2}}} \right)^4 \leq \\
 & \leq \frac{(n-1)^{2(n-1)}}{(n-1)^{2(n-1)}} \cdot \left( \prod_{i=1}^n a_i \right)^2 \cdot \left( \sum_{i=1}^n a_i^2 \right)^2 \cdot \frac{n \cdot (\sum_{i=1}^n a_i^2)}{(\prod_{i=1}^n a_i)^2} = n \cdot \left( \sum_{i=1}^n a_i^2 \right)^3
 \end{aligned}$$

**674.** If  $a, b, c > 0, a^2b^2 + b^2c^2 + c^2a^2 = 12abc$  then:

$$\sqrt[3]{\frac{a}{4a+bc}} + \sqrt[3]{\frac{b}{4b+ca}} + \sqrt[3]{\frac{c}{4c+ab}} \geq \frac{3}{2}$$

*Proposed by Daniel Sitaru-Romania*



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**Solution 1 by George Florin Șerban-Romania**

$$a^2b^2 + b^2c^2 + c^2a^2 = 12abc \Rightarrow \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = 12$$

$$\text{Let } x = \frac{ab}{c}, y = \frac{bc}{a}, z = \frac{ca}{b} \Rightarrow x + y + z = 12$$

$$\sum_{cyc} \sqrt[3]{\frac{a}{4a+bc}} = \sum_{cyc} \frac{1}{\sqrt[3]{4+\frac{bc}{a}}} = \sum_{cyc} \frac{1}{\sqrt[3]{4+x}} \stackrel{(*)}{\geq} \frac{3}{2}$$

*Let be the function:  $f: (0, \infty) \rightarrow \mathbb{R}, f(x) = \frac{1}{\sqrt[3]{4+x}}$  –convex.*

$$f'(x) = -\frac{1}{3}(x+4)^{-\frac{4}{3}}, f''(x) = \frac{4}{9}(x+4)^{-\frac{7}{3}} > 0$$

*From Jensen Inequality, we have:*

$$\begin{aligned} f\left(\frac{x+y+z}{3}\right) &\leq \frac{f(x)+f(y)+f(z)}{3} \Leftrightarrow \\ \sum_{cyc} \frac{1}{\sqrt[3]{4+x}} &\geq \frac{3}{\sqrt[3]{4+\frac{1}{3}\sum_{cyc} x}} \Leftrightarrow \sum_{cyc} \frac{1}{\sqrt[3]{4+x}} \geq \frac{3}{2} \end{aligned}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\sum_{cyc} \sqrt[3]{\frac{abc}{4abc+b^2c^2}} \stackrel{AM-GM}{\geq} 3 \sqrt[3]{\prod_{cyc} \sqrt[3]{\frac{abc}{4abc+b^2c^2}}} = 3 \sqrt[3]{\frac{abc}{\prod_{cyc} \sqrt[3]{4abc+b^2c^2}}} = \Omega$$

*Other,*

$$\begin{aligned} \prod_{cyc} \sqrt[3]{4abc+b^2c^2} &= \sqrt[3]{\prod_{cyc} (4abc+b^2c^2)} \stackrel{AM-GM}{\leq} \frac{\sum_{cyc} (4abc+b^2c^2)}{3} = \\ &= \frac{12abc + \sum_{cyc} b^2c^2}{3} = \frac{12abc + 12abc}{3} = 8abc \end{aligned}$$

$$\text{Therefore, } \Omega \geq 3 \sqrt[3]{\frac{abc}{8abc}} = \frac{3}{2}$$

$$\sum_{cyc} \sqrt[3]{\frac{abc}{4abc+b^2c^2}} \geq 3/2$$



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**675. If  $x, y, z \geq 1$  then:**

$$\left( \prod_{cyc} (x+1) + 8xyz \right) \prod_{cyc} (x+3) \geq 16 \prod_{cyc} (3x+1)$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

$$x, y, z \geq 1 \Rightarrow x+3, y+3, z+3 \geq 4 \Rightarrow \prod_{cyc} (x+3) \geq 4 \cdot 4 \cdot 4$$

*So, we must show that:*

$$4 \left( \prod_{cyc} (x+1) + 8xyz \right) \geq \prod_{cyc} (3x+1) \Leftrightarrow$$

$$4(9xyz + xy + yz + zx + x + y + z + 1) \geq (27xyz + 9(xy + yz + zx) + 3(x + y + z) + 1)$$

$$9xyz \geq 5(xy + yz + zx) - (x + y + z) - 3 \Leftrightarrow$$

$$9xyz + (x + y + z) + 3 \geq 5(xy + yz + zx)$$

$$\text{Let } f(z) = (9xy + 1 - 5(x + y))z + x + y - 5xy + 3; (z \geq 1)$$

$$\begin{aligned} 9xy + 1 - 5(x + y) &= (9x - 5)y + 1 - 5x \geq (9x - 5) \cdot 1 + 1 - 5x = \\ &= 4x + 1 - 5 \geq 4 \cdot 1 + 1 - 5 = 0; (x, y \geq 1) \end{aligned}$$

$$f(z) \nearrow [1, \infty) \Rightarrow f(z) \geq f(1) = 4(x-1)(y-1) \geq 0, \forall x, y \geq 1$$

$$f(z) \geq 0 \Rightarrow 9xyz + (x + y + z) + 3 \geq 5(xy + yz + zx), \forall x, y \geq 1$$

**Solution 2 by Michael Sterghiou-Greece**

$$\left( \prod_{cyc} (x+1) + 8xyz \right) \prod_{cyc} (x+3) \geq 16 \prod_{cyc} (3x+1); (1)$$

$$\text{Let } (p, q, r) = (\sum_{cyc} x, \sum_{cyc} xy, \prod_{cyc} x)$$

*After some computation ( $p \geq 3, q \geq 3, r \geq 1$ ) then (1) becomes:*

$$9p^2 + 12pq + 82pr - 12p + 3q^2 + 28qr - 114q - 188r + 9r^2 + 11 \geq 0; (2)$$

*As  $\sum_{cyc} x^2 = p^2 - 2q$ ;  $\sum_{cyc} x^2 y^2 = q^2 - 2pr$ . Note that  $pr \geq q$ .*

*Assume WLOG  $x = \max\{x, y, z\}$ :*



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$$(x+y+z)xyz \geq x(x+y+z) = x^2 + xy + zx \geq yz + xy + zx = q$$

*Now, we can write (2) as:*

$$5(p^2 - 3q) + 4p(p-3) + 3q(q-3) + 12q(p-3) + 54(pr-q) + r(28q+28p - 188)r + 9r^2 + 11 \geq 0 \text{ which reduces to:}$$

$$28r(p+q-6) - 20r + 9r^2 + 11 \geq 0 \text{ as all terms are positive or zero.}$$

*Given that  $p+q \geq 2\sqrt{pq} \geq 6\sqrt{r}$ ; ( $pq \geq 9r$ ) the last inequality reduces to:*

$$(\sqrt{r}-1)(9\sqrt[3]{r^2} + 177r - 11\sqrt{r} - 11) \geq 0 \text{ for } r \geq 1. \text{ Done!}$$

*Equality for  $p=3, r=1, q=3 \Leftrightarrow x=y=z=1$ .*

**Solution 3 by Michael Sterghiou-Greece**

$$\left( \prod_{cyc} (x+1) + 8xyz \right) \prod_{cyc} (x+3) \geq 16 \prod_{cyc} (3x+1); (1)$$

$$\text{Let } f(x, y, z) = (\prod_{cyc} (x+1) + 8xyz) \prod_{cyc} (x+3) - 16 \prod_{cyc} (3x+1)$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \left( \prod_{cyc} (x+3) \right) (9yz + y + z + 1) + (y+3)(z+3) \left( \prod_{cyc} (x+1) + 8xyz \right) \\ &\quad - 48(3y+1)(3z+1) \end{aligned}$$

$$\frac{\partial^2 f}{\partial x^2} = 2(y+3)(z+3)(9yz + y + z + 1) > 0$$

*Hence  $\frac{\partial f}{\partial x} >$  with fixed  $y, z$ :  $f'(x) >$  then*

$$\begin{aligned} f'(x) \geq f'(1) &= 2(23y^2z^2 + 72y^2z + 9y^2 + 72yz^2 + 12yz - 36y + 9z^2 - 36z + 3) = \\ &g(y) \text{ with } z - \text{fixed.} \end{aligned}$$

*Now,  $g'(y) > 0$ ; ( $x, y \geq 1$ )  $\Rightarrow g(y) > g(1)$ ; ( $z \geq 1$ ) and therefore  $f'(x) > 0$  or*

$$f(x) > f(1) = 8[(5z^2 + 16z + 3)y^2 + 4(4z^2 - 5z - 3)y + 9z^2 - 12z + 1] = h(y)$$

$$\text{Again } \frac{1}{8}h''(y) = 10z^2 + 32z + 6 > 0 \text{ and } h'(y) \geq h'(1) > 0$$

*So,  $h(y) \geq h(1) = 8(z-1)(3z+1) > 0$  and  $f(x) \geq 0$ .*

*Equality for  $x = y = z = 1$ . Done!*

**Solution 4 by Pavlos Trifon-Greece**

$$\begin{aligned}
 & \left( \prod_{cyc} (x+1) + 8xyz \right) \prod_{cyc} (x+3) \geq 16 \prod_{cyc} (3x+1) \Leftrightarrow \\
 & \prod_{cyc} (x+1) \prod_{cyc} (x+3) + 8xyz \prod_{cyc} (x+3) \geq 16 \prod_{cyc} (3x+1) \Leftrightarrow \\
 & \prod_{cyc} (x+1)(x+3) + 8 \prod_{cyc} x(x+3) \geq 16 \prod_{cyc} (3x+1) \Leftrightarrow \\
 & \prod_{cyc} (x^2 + 4x + 3) + 8 \prod_{cyc} (x^2 + 3x) \geq 16 \prod_{cyc} (3x+1) \Leftrightarrow \\
 & \prod_{cyc} \left( \frac{x^2 + 4x + 3}{3x+1} \right) + 8 \prod_{cyc} \left( \frac{x^2 + 3x}{3x+1} \right) \geq 16
 \end{aligned}$$

Let:  $f(x) = \frac{x^2 + 4x + 3}{3x+1}$ ,  $f'(x) = \frac{3x^2 + 2x - 5}{9x^2 + 6x + 1}$ ,  $f'(1) = 0$ ,  $f'(x) > 0$ ,  $\forall x > 1 \Rightarrow f_{min} = f(1) = 2$

And  $g(x) = \frac{x^2 + 3x}{3x+1}$ ,  $x \geq 1$ ,  $g_{min} = g(1) = 1$ . Therefore,

$$\prod_{cyc} f(x) + 8 \prod_{cyc} g(x) \geq f^3(1) + 8 \cdot g^3(1) = 8 + 8 \cdot 1 = 16$$

Equality holds for  $x = y = z = 1$ .

**676. If  $a, b > 0$  then:**

$$256 \sqrt{\frac{a^2 + b^2}{2}} \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} \right)^3 \leq 27 \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \right)^4$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Michael Sterghiou-Greece**

$$256 \sqrt{\frac{a^2 + b^2}{2}} \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} \right)^3 \leq 27 \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \right)^4 ; (1)$$

$$\text{Let: } t = \sqrt{\frac{a^2 + b^2}{2}} \text{ and } u = \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}$$

Then (1) becomes:

$$256tu^3 - 27(t+u)^4 \leq 0 \Leftrightarrow -(u-3t)^2(27u^2 + 14ut + 3t^2) \leq 0$$



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*Which is true for  $u, t > 0$ . Done!*

**Solution 2 by Soumava Chakraborty-Kolkata-India**

Let  $\sqrt{\frac{a^2 + b^2}{2}} = Q, \frac{a+b}{2} = A, \sqrt{ab} = G, \frac{2ab}{a+b} = H$  and  $\because Q \geq A \geq G \geq H \therefore 3Q \geq A + G + H = m$  (say)

$$\begin{aligned} & \text{Now, } 256 \sqrt{\frac{a^2 + b^2}{2}} \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} \right)^3 \\ & \leq 27 \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \right)^4 \Leftrightarrow 256Qm^3 \stackrel{(i)}{\geq} 27(Q+m)^4 \end{aligned}$$

Now,  $4 \cdot 3Q \cdot m \leq (3Q + m)^2$  (equality when  $3Q = m$ )  $\Rightarrow 12Qm \leq (3Q + m)^2$

$$\begin{aligned} & \Rightarrow 12Qm \cdot \frac{256m^2}{12} \leq \frac{256m^2}{12} \cdot (3Q + m)^2 \stackrel{?}{\geq} 27(Q+m)^4 \\ & \Leftrightarrow 81(Q+m)^4 \stackrel{?}{\geq} 64m^2(3Q+m)^2 \Leftrightarrow 9(Q+m)^2 \stackrel{?}{\geq} 8m(3Q+m) \\ & \Leftrightarrow 9Q^2 + 9m^2 + 18Qm \stackrel{?}{\geq} 24Qm + 8m^2 \Leftrightarrow 9Q^2 + m^2 - 6Qm \stackrel{?}{\geq} 0 \\ & \Leftrightarrow (3Q - m)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true (equality when } 3Q = m \Leftrightarrow a = b) \Rightarrow 256Qm^3 \\ & \leq 27(Q+m)^4 \Rightarrow (i) \text{ is true} \end{aligned}$$

$$\Rightarrow 256 \sqrt{\frac{a^2 + b^2}{2}} \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} \right)^3 \leq 27 \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \right)^4$$

(equality when  $a = b$ ) (Proved)

**Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand**

$$\begin{aligned} & 256 \sqrt{\frac{a^2 + b^2}{2}} \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} \right)^3 \leq 27 \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \right)^4 \\ & 256 \sqrt{\frac{a^2 + b^2}{2}} \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} \right)^3 \leq \end{aligned}$$



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$$\leq 4^4 \left( \sqrt{\frac{a^2 + b^2}{2}} + \frac{3 \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} \right)}{3} \right)^4$$

$$\leq \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \right)^4$$

$$\left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \right)^4 \leq \left( \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \right)^4$$

$$\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}} \leq \frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}}$$

**677. If  $a_i > 0, i = 1, n, n \in \mathbb{N}, n \geq 2$  then prove:**

$$\sum_{i=1}^n \frac{a_i}{(\sum_{i \neq i} a_j)^n} \geq \left( \frac{n}{n-1} \right)^n \cdot \left( \sum_{i=1}^n a_i \right)^{1-n}$$

*Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan*

**Solution by Florică Anastase-Romania**

$$\begin{aligned} \sum_{i=1}^n \frac{a_i}{(\sum_{i \neq i} a_j)^n} &\stackrel{\text{Chebishev's}}{\geq} \frac{1}{n} \cdot \sum_{i=1}^n a_i \cdot \sum_{i=1}^n \frac{1}{(\sum_{i \neq i} a_j)^n} \stackrel{\text{Radon}}{\geq} \\ &\stackrel{\text{Radon}}{\geq} \frac{1}{n} \cdot \frac{(1+1+\dots+1)^{n+1}}{(\sum_{i=1}^n (\sum_{i \neq i} a_j))^n} = \frac{1}{n} \cdot \sum_{i=1}^n a_i \cdot \frac{n^{n+1}}{[(n-1)\sum_{i=1}^n a_i]^n} = \\ &= \left( \frac{n}{n-1} \right)^n \cdot \frac{1}{(\sum_{i=1}^n a_i)^{n-1}} = \left( \frac{n}{n-1} \right)^n \cdot \left( \sum_{i=1}^n a_i \right)^{1-n} \end{aligned}$$

**678. If  $a, b, c, d > 0$  then:**

$$3^6 (abcd)^2 \left( \sum_{cyc} \frac{a}{\sqrt{(b+c+d)^3}} \right)^4 \leq 4 \left( \sum_{cyc} a^2 \right)^3$$

*Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan*



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**Solution by Tran Hong-Dong Thap-Vietnam**

*By AM-GM inequality, we have:*

$$\begin{aligned}
 b + c + d &\geq 3\sqrt[3]{bcd} \quad (\text{and analogs}) \\
 \Rightarrow (b + c + d)^3 &\geq 3^3 \cdot bcd \quad (\text{and analogs}) \\
 \Rightarrow \sqrt{(b + c + d)^3} &\geq \sqrt{3^3 \cdot bcd} \quad (\text{and analogs}) \\
 \Rightarrow \sum_{cyc} \frac{a}{\sqrt{(b + c + d)^3}} &\leq \sum_{cyc} \frac{a}{\sqrt{3^3 \cdot bcd}} \stackrel{BCS}{\leq} \sqrt{\left(\sum a^2\right) \left(\sum \frac{1}{3^3 bcd}\right)} \\
 \left(\sum_{cyc} \frac{a}{\sqrt{(b + c + d)^3}}\right)^4 &\leq \left(\sqrt{\left(\sum a^2\right) \left(\sum \frac{1}{3^3 bcd}\right)}\right)^4 = \frac{1}{3^6} \left(\sum \frac{1}{bcd}\right)^2 \left(\sum a^2\right)^2 = \\
 = \frac{1}{3^6} \left(\frac{\sum a}{abcd}\right)^2 \left(\sum a^2\right)^2 &= \frac{(\sum a)^2 (\sum a^2)^2}{3^6 (abcd)^2} \stackrel{(\sum a)^2 \leq 3 \sum a^2}{\leq} \frac{3 (\sum a^2)^3}{3^6 (abcd)^2} = \frac{(\sum a^2)^3}{3^5 (abcd)^2} \\
 \Rightarrow 3^6 (abcd)^2 \left(\sum_{cyc} \frac{a}{\sqrt{(b + c + d)^3}}\right)^4 &\leq 3 \left(\sum a^2\right)^3 < 4 \left(\sum a^2\right)^3 \quad (\text{true})
 \end{aligned}$$

*Because:  $4(\sum a^2)^3 - 3(\sum a^2)^3 = (\sum a^2)^3 > 0, \forall a, b, c > 0$ .*

**679. If  $x, y, z > 0, xyz = 1$  then:**

$$\frac{1}{x^5(y+z)} + \frac{1}{y^5(z+x)} + \frac{1}{z^5(x+y)} \geq \frac{3}{2}$$

*Proposed by Rajeev Rastogi-India*

**Solution 1 by Abdul Hannan-Tezpur-India**

*Let  $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$  then  $abc = 1$  and the desired inequality becomes:*

$$\sum_{cyc} \frac{a^5}{\frac{1}{b} + \frac{1}{c}} = \sum_{cyc} \frac{a^5 bc}{b+c} = \sum_{cyc} \frac{a^4}{b+c} \geq \frac{3}{2}$$

*This is true, because:*

$$\sum_{cyc} \frac{a^4}{b+c} \stackrel{\text{Chebyshev's}}{\geq} \frac{1}{3} \left( \sum_{cyc} a^3 \right) \left( \sum_{cyc} \frac{a}{b+c} \right) \stackrel{\substack{\text{AM-GM} \\ \text{Nesbitt}}}{\geq} abc \cdot \frac{3}{2} = \frac{3}{2}$$



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**Solution 2 by Florică Anastase-Romania**

$$\begin{aligned}
 \sum_{cyc} \frac{1}{x^5(y+z)} &= \sum_{cyc} \frac{x^5y^5}{y+z} = \sum_{cyc} \frac{x^4y^4}{\frac{1}{y} + \frac{1}{z}} \stackrel{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} x^2y^2\right)^2}{2 \sum_{cyc} \frac{1}{x}} \stackrel{xyz=1}{=} \\
 &= \frac{(\sum_{cyc} x^2y^2)(\sum_{cyc} x^2y^2)}{2 \sum_{cyc} xy} \stackrel{\text{AM-GM}}{\geq} \frac{3xyz\sqrt[3]{xyz}(\sum_{cyc} x^2y^2)}{2 \sum_{cyc} xy} = \\
 &= \frac{3(\sum_{cyc} x^2y^2)}{2 \sum_{cyc} xy} \geq \frac{3}{2} \\
 \Leftrightarrow \left( \sum_{cyc} x^2y^2 \right) &\geq \sum_{cyc} xy \Leftrightarrow \frac{1}{2} \sum_{cyc} (xy - xz)^2 \geq 0
 \end{aligned}$$

**Solution 3 by Ram Vijay Singh Rathmore-India**

$$\begin{aligned}
 x, y, z > 0; xyz &= 1 \\
 \sqrt[3]{(xyz)^5(x+y)(y+z)(z+x)} &\stackrel{\text{GM-HM}}{\geq} \frac{3}{\sum_{cyc} \frac{1}{x^5(y+z)}} \\
 \Leftrightarrow \sum_{cyc} \frac{1}{x^5(y+z)} &\geq \frac{3}{\sqrt[3]{(x+y)(y+z)(z+x)}} \\
 \text{Also, } (x+y)(y+z)(z+x) &\stackrel{\text{GM-AM}}{\geq} 2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx} = 8
 \end{aligned}$$

*Therefore,*

$$\sum_{cyc} \frac{1}{x^5(y+z)} \geq \frac{3}{2}$$

**Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand**

*For  $x, y, z > 0$  and  $xyz = 1$  we have:*

$$\begin{aligned}
 \frac{1}{x^5(y+z)} + \frac{1}{y^5(z+x)} + \frac{1}{z^5(x+y)} &= \frac{\frac{1}{x^5}}{y+z} + \frac{\frac{1}{y^5}}{z+x} + \frac{\frac{1}{z^5}}{x+y} \geq \\
 &\geq \frac{\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^5}{3^3 \cdot 2(x+y+z)} \geq \frac{3}{2}
 \end{aligned}$$



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$$\Leftrightarrow \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)^5 \geq 3^4(x+y+z) \Leftrightarrow \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)^2 \geq 3(x+y+z)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq x+y+z = \frac{2}{yz} + \frac{2}{zx} + \frac{2}{xy} = 2(x+y+z) \text{ true, because}$$

$$x^2y^2 + y^2z^2 + z^2x^2 \geq xy^2z + yz^2x + zx^2y = x+y+z$$

**Solution 5 by Abner Chinga Bazo-Lima-Peru**

$$\begin{aligned} \frac{1}{x^5(y+z)} + \frac{1}{y^5(z+x)} + \frac{1}{z^5(x+y)} &= \frac{\left(\frac{1}{x^2}\right)^2}{xy+zx} + \frac{\left(\frac{1}{y^2}\right)^2}{zy+xy} + \frac{\left(\frac{1}{z^2}\right)^2}{xz+yz} \geq \\ &\geq \frac{\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)^2}{2(xy+yz+zx)} \stackrel{xyz=1}{\implies} \\ \frac{1}{x^5(y+z)} + \frac{1}{y^5(z+x)} + \frac{1}{z^5(x+y)} &\geq \frac{\left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right)^2}{2(xy+yz+zx)} = \\ &= \frac{\left(\frac{x+y+z}{xyz}\right)^2}{2(xy+yz+zx)} = \frac{(x+y+z)^2}{2(xy+yz+zx)} \geq \frac{3}{2} \\ \frac{1}{x^5(y+z)} + \frac{1}{y^5(z+x)} + \frac{1}{z^5(x+y)} &\geq \frac{3}{2} \end{aligned}$$

**680. If  $a, b, c > 0, abc \leq 1, \lambda \geq 0, n \in \mathbb{N}, n \geq 2$  then:**

$$\frac{1}{a^n(b+\lambda c)} + \frac{1}{b^n(c+\lambda a)} + \frac{1}{c^n(a+\lambda b)} \geq \frac{3}{\lambda+1}$$

*Proposed by Marin Chirciu-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

For  $x, y, z \geq 0, \lambda \geq 0$  ( $\lambda$  - fixed) we have:

$$\frac{x}{z+\lambda y} + \frac{y}{x+\lambda z} + \frac{z}{y+\lambda x} \geq \frac{3}{1+\lambda}; \quad (1)$$

*Proof:*

$$\frac{x}{z+\lambda y} + \frac{y}{x+\lambda z} + \frac{z}{y+\lambda x} = \sum_{cyc} \frac{x^2}{xz+\lambda x} \stackrel{\text{Bergstrom}}{\geq} \frac{(x+y+z)^2}{(1+\lambda)(xy+yz+zx)} \geq \frac{3}{1+\lambda}$$



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$(Because (x+y+z)^2 \geq 3(xy+yz+zx)) \Rightarrow (1) - true.$

*Now, if  $n = 2$  then:*

$$\frac{1}{a^2(b+\lambda c)} + \frac{1}{b^2(c+\lambda a)} + \frac{1}{c^2(a+\lambda b)} \underset{\text{cyc}}{=} \sum \frac{1}{ab+\lambda ac} \stackrel{\frac{1}{a} \geq bc}{\geq} \sum \frac{bc}{ab+\lambda ac} \geq \frac{3}{1+\lambda}$$

*(Because: in (1) choose  $x = bc, y = ca, z = ab$ )*

*If  $n \geq 3, n \in \mathbb{N}$  and  $0 < abc \leq 1 \Rightarrow \exists \alpha \geq c > 0$  such that  $aba = 1$ .*

$$\begin{aligned} \sum_{\text{cyc}} \frac{1}{a^n(b+\lambda c)} &\geq \sum_{\text{cyc}} \frac{1}{a^n(b+\lambda \alpha)} = \sum_{\text{cyc}} \frac{\left(\frac{1}{a}\right)^{n-1}}{ab+\lambda a\alpha} \stackrel{\frac{1}{a}=ba}{\geq} \sum_{\text{cyc}} \frac{(ba)^{n-1}}{ab+\lambda a\alpha} \stackrel{\text{Holder}}{\geq} \\ &\stackrel{\text{Holder}}{\geq} \frac{(ab+ba+\alpha a)^{n-1}}{3^{n-3} \cdot (1+\lambda)(ab+ba+\alpha a)} = \frac{1}{1+\lambda} \cdot \frac{(ab+ba+\alpha a)^{n-1}}{3^{n-3} \cdot (ab+ba+\alpha a)} \stackrel{(2)}{\geq} \frac{3}{1+\lambda} \\ (2) \Leftrightarrow (ab+ba+\alpha a)^{n-1} &\geq 3^{n-2}(ab+ba+\alpha a) \end{aligned}$$

*Which is clearly true, because:  $t = ab + ba + \alpha a \geq 3\sqrt[3]{aba} = 3$*

*$t^{n-1} \geq 3^{n-2}t \Leftrightarrow t(t^{n-2} - 3^{n-2}) \geq 0$  true by  $t \geq 3 \Rightarrow t^{n-2} \geq 3^{n-2}$ .*

*Equality holds if and only if  $t = 3 \Leftrightarrow a = b = c = \alpha = 1$ . Proved.*

**681. If  $x, y, z: 0$  such that  $xyz = 1$  and  $n \in \mathbb{N}, n \geq 2$  then:**

$$\frac{1}{(x+1)^n} + \frac{1}{(y+1)^n} + \frac{1}{(z+1)^n} \geq \frac{3}{2^n}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

*Lemma: If  $a_1, a_2, \dots, a_n \in (0, \infty), n \geq 3$  such that  $\sqrt[n]{a_1 a_2 \cdots a_n} = p \geq \sqrt{n-1}$  then:*

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \geq \frac{n}{(1+p)^2}; (1)$$

*In (1) choosing:  $n = 3, p = 1, a_1 = x, a_2 = y, a_3 = z, xyz = 1$*

$$\Rightarrow \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} \geq \frac{3}{4}; (2)$$

*Now, by AM-GM inequality, we have:*



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$$\begin{aligned} \frac{1}{(x+1)^n} + \frac{1}{(x+1)^n} + \underbrace{\frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{(n-2)\text{ times}} &\geq n \sqrt[n]{\left(\frac{1}{(x+1)^n}\right)^2 \cdot \left(\frac{1}{2^n}\right)^{n-2}} = \frac{1}{(x+1)^2} \cdot \frac{n}{2^{n-2}} \\ \Rightarrow \frac{2}{(x+1)^n} + \frac{n-2}{2^n} &\geq \frac{1}{(x+1)^2} \cdot \frac{n}{2^{n-2}}; \quad (3) \end{aligned}$$

*Similarly:*

$$\frac{2}{(y+1)^n} + \frac{n-2}{2^n} \geq \frac{1}{(y+1)^2} \cdot \frac{n}{2^{n-2}}; \quad (4)$$

$$\frac{2}{(z+1)^n} + \frac{n-2}{2^n} \geq \frac{1}{(z+1)^2} \cdot \frac{n}{2^{n-2}}; \quad (5)$$

*From (3) + (4) + (5) we have:*

$$\begin{aligned} \frac{2}{(x+1)^n} + \frac{2}{(y+1)^n} + \frac{2}{(z+1)^n} + \frac{3(n-2)}{2^n} &\geq \frac{n}{2^{n-2}} \left( \frac{1}{(x+1)^2} + \frac{1}{(y+1)^2} + \frac{1}{(z+1)^2} \right) \geq \\ &\stackrel{(2)}{\geq} \frac{n}{2^{n-2}} \cdot \frac{3}{4} = \frac{3n}{2^n} \\ \Rightarrow \frac{2}{(x+1)^n} + \frac{2}{(y+1)^n} + \frac{2}{(z+1)^n} &\geq \frac{3n}{2^n} - \frac{3(n-2)}{2^n} = \frac{3 \cdot 2}{2^n} \\ \frac{1}{(x+1)^n} + \frac{1}{(y+1)^n} + \frac{1}{(z+1)^n} &\geq \frac{3}{2^n} \end{aligned}$$

**682. If  $a, b, c > 0$  then:**

$$(a^3 + b^3 + c^3)^2 \cdot \left( \frac{1}{abc} + \frac{4}{(a+b)(b+c)(c+a)} \right)^2 \geq \frac{243}{4}$$

*Proposed by Pavlos Trifon-Greece*

**Solution by Michael Sterghiou-Greece**

$$(a^3 + b^3 + c^3)^2 \cdot \left( \frac{1}{abc} + \frac{4}{(a+b)(b+c)(c+a)} \right)^2 \geq \frac{243}{4}; \quad (1)$$

(1) – is homogeneous so WLOG let  $P = \sum_{cyc} a$ ,

$$q = \sum_{cyc} ab, r = abc, q \leq 3 (\text{by AM - GM})$$

$$(1) \text{ becomes: } (9 - 2q)^3 \left( \frac{1}{2r} + \frac{1}{2r} + \frac{4}{3q-r} \right)^2 \geq \frac{243}{4}$$



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(as  $\prod_{cyc}(a+b) = pq - r = 3q - r$ ) which reduces by AM-GM to the stronger

$$\text{inequality: } (9 - 2q)^2 \left( 3 \sqrt[3]{\frac{4}{4r^2(3q-r)}} \right)^2 - \frac{243}{4} \geq 0 \text{ or}$$

$$4(9 - 2q)^2 - 27 \sqrt[3]{[r^2(3q - r)]^2} \geq 0; \quad (2)$$

*Now,*

$f(r) = r^2(3q - r)$  is increasing function of  $r$ , ( $f'(r) = 3r(2q - 1) > 0$ ) and  $r \leq \frac{q}{3} \sqrt{\frac{q}{3}}$ .

$$\text{So, (2) reduces to } 4(9 - 2q)^3 - 3q^2 \sqrt[3]{\left(\frac{q}{3} \sqrt{\frac{q}{3}}\right)^2} \geq 0.$$

Now,  $g(q) = \frac{q}{3} \sqrt{\frac{q}{3}}$  is increasing function of  $q$  for  $q \leq 3$ .

*So, it is enough that:*

$$4(9 - 2q)^3 - 3 \cdot 9 \sqrt[3]{(9 - 1)^2} \geq 0 \text{ which reduces to } q \leq 3, \text{ true. Done!}$$

**683. If  $a, b, c > 0$ ,  $\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} = \frac{3}{4}$  then:**

$$16 \sum_{cyc} \frac{\sqrt{ab}}{a+b} + \sum_{cyc} \frac{(a+b)^2}{ab} \geq 12 + 4 \sum_{cyc} \frac{a+b}{\sqrt{ab}}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

For  $a, b, c > 0$  and  $\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} = \frac{3}{4}$ . Inequality becomes as:

$$16 \sum_{cyc} \frac{\sqrt{ab}}{a+b} + \sum_{cyc} \frac{(a+b)^2}{ab} \geq 16 \sum_{cyc} \frac{ab}{(a+b)^2} + 4 \sum_{cyc} \frac{a+b}{\sqrt{ab}}; \quad (1)$$

*Hence, we must that:*

$$16 \cdot \frac{\sqrt{ab}}{a+b} + \frac{(a+b)^2}{ab} \geq 16 \cdot \frac{ab}{(a+b)^2} + 4 \cdot \frac{a+b}{\sqrt{ab}};$$

$$\Leftrightarrow \frac{16v}{u} + \frac{u^2}{v^2} \geq \frac{16v^2}{u^2} + \frac{4u}{v}; (\because u = a+b; v = \sqrt{ab}; u \geq 2v > 0)$$



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$$\begin{aligned}
 &\Leftrightarrow 16uv^3 + u^4 \geq 16v^4 + 4vu^3 \Leftrightarrow u^4 - 16v^4 - 4uv(u^2 - 4v^2) \geq 0; \\
 &\Leftrightarrow (u^2 + 4v^2)(u^2 - 4v^2) - 4uv(u^2 - 4v^2) \geq 0 \Leftrightarrow (u^2 - 4v^2)(u^2 + 4v^2 - 4uv) \geq 0; \\
 &\Leftrightarrow (u^2 - 4v^2)(u - 2v)^2 \geq 0 \Leftrightarrow (u - 2v)^3(u + 2v) \geq 0;
 \end{aligned}$$

*Which is true by:  $u \geq 2v > 0$*

*$\Rightarrow (1)$  true. Proved. Equality  $\Leftrightarrow u = 2v \Leftrightarrow a = b = c = 1.$*

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

$$\begin{aligned}
 16 \left( \frac{\sqrt{ab}}{a+b} + \frac{\sqrt{bc}}{b+c} + \frac{\sqrt{ca}}{c+a} \right) &\geq 16 \left( \frac{\sqrt{ab}(a+b)}{(a+b)^2} + \frac{\sqrt{bc}(b+c)}{(b+c)^2} + \frac{\sqrt{ca}(c+a)}{(c+a)^2} \right) \geq \\
 &\geq 16 \cdot 2 \left( \frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} \right) = 32 \cdot \frac{3}{4} = 24 \\
 4 + \frac{(a+b)^2}{ab} &\geq \frac{4(a+b)}{\sqrt{ab}} \Leftrightarrow 4ab + (a+b)^2 \geq 4(a+b)\sqrt{ab} \\
 &\Leftrightarrow 4(a+b)\sqrt{ab} \geq 4(a+b)\sqrt{ab} (\text{true}) \\
 &\text{Similarly:} \\
 4 + \frac{(b+c)^2}{bc} &\geq \frac{4(b+c)}{\sqrt{bc}}, \quad 4 + \frac{(c+a)^2}{ca} \geq \frac{4(c+a)}{\sqrt{ca}} \\
 &\text{Hence} \\
 16 \sum_{cyc} \frac{\sqrt{ab}}{a+b} + \sum_{cyc} \frac{(a+b)^2}{ab} &\geq 12 + 4 \sum_{cyc} \frac{a+b}{\sqrt{ab}}
 \end{aligned}$$

**684. If  $0 < m \leq a_i \leq M, i \in \overline{1, n}, n \in \mathbb{N}, n > 2$  then:**

$$\sum_{i=1}^n \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j \neq i}}^n a_j^2}} \leq \frac{n}{2\sqrt{n-1}} \left( \frac{m}{M} + \frac{M}{m} \right)$$

*Proposed by Seyran Ibrahimov-Masilli-Azerbaijan*

**Solution 1 by Michael Sterghiou-Greece**

$$\sum_{i=1}^n \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j \neq i}}^n a_j^2}} \leq \frac{n}{2\sqrt{n-1}} \left( \frac{m}{M} + \frac{M}{m} \right); \quad (1)$$

$f(t) = \sqrt{t}$  is concave on  $(0, \infty)$



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$$\sum_{i=1}^n \sqrt{\frac{a_i^2}{x-a_i^2}} \leq n \cdot \sqrt{\frac{\left(\frac{p}{n}\right)^2}{x-\left(\frac{p}{n}\right)^2}}; \quad (2)$$

Where  $x = \sum_{i=1}^n a_i^2$  and  $p = \sum_{i=1}^n a_i$  with  $x \geq \frac{p^2}{n}$ .

$$(2) \Rightarrow LHS_{(1)} \stackrel{\text{Jensen}}{\leq} n \cdot \sqrt{\frac{p^2}{n^2 x - p^2}} \leq n \cdot \sqrt{\frac{p^2}{n^2 \cdot \frac{p^2}{n} - p^2}} = \frac{n}{\sqrt{n-1}}; \quad (3)$$

But  $\frac{m}{M} + \frac{M}{m} \stackrel{AM-GM}{\geq} 2 \Rightarrow \frac{1}{2} \left( \frac{m}{M} + \frac{M}{m} \right) \geq 1$  and from (3)

Since  $LHS_{(1)} \leq \frac{n}{\sqrt{n-1}}$  it follows that:

$$\sum_{i=1}^n \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j \neq i}}^n a_j^2}} \leq \frac{n}{2\sqrt{n-1}} \left( \frac{m}{M} + \frac{M}{m} \right)$$

**Solution 2 by Florică Anastase-Romania**

Let  $S = \sum_{i=1}^n a_i$

From Cauchy-Schwartz Inequality we have:

$$\begin{cases} a_1^2 + a_2^2 + \dots + a_{n-1}^2 \geq \frac{(a_1 + a_2 + \dots + a_{n-1})^2}{n-1} \\ \dots \\ a_2^2 + a_3^2 + \dots + a_n^2 \geq \frac{(a_2 + a_3 + \dots + a_n)^2}{n-1} \end{cases} \Rightarrow$$

$$\begin{cases} \frac{a_n}{\sqrt{a_1^2 + a_2^2 + \dots + a_{n-1}^2}} \leq \frac{a_n \sqrt{n-1}}{S-a_n} \\ \dots \\ \frac{a_1}{\sqrt{a_2^2 + a_3^2 + \dots + a_n^2}} \leq \frac{a_1 \sqrt{n-1}}{S-a_1} \end{cases}$$

$$\sum_{i=1}^n \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j \neq i}}^n a_j^2}} \leq \sqrt{n-1} \left( \frac{a_1}{S-a_1} + \frac{a_2}{S-a_2} + \dots + \frac{a_n}{S-a_n} \right) \stackrel{(1)}{\leq} \frac{n}{\sqrt{n-1}}$$



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$$\begin{aligned}
 n \left( \frac{a_1}{S-a_1} + \frac{a_2}{S-a_2} + \cdots + \frac{a_n}{S-a_n} \right) &\leq \frac{S}{S-a_1} + \frac{S}{S-a_2} + \cdots + \frac{S}{S-a_n} \Leftrightarrow \\
 (n-1) \left( \frac{a_1}{S-a_1} + \frac{a_2}{S-a_2} + \cdots + \frac{a_n}{S-a_n} \right) &\leq \frac{S-a_1}{S-a_1} + \frac{S-a_2}{S-a_2} + \cdots + \frac{S-a_n}{S-a_n} \Leftrightarrow \\
 \frac{a_1}{S-a_1} + \frac{a_2}{S-a_2} + \cdots + \frac{a_n}{S-a_n} &\leq \frac{n}{n-1} \Rightarrow (1) \text{ true.}
 \end{aligned}$$

$$\text{But } \frac{m}{M} + \frac{M}{m} \stackrel{\text{AM-GM}}{\geq} 2 \Rightarrow \frac{1}{2} \left( \frac{m}{M} + \frac{M}{m} \right) \geq 1; \quad (2)$$

*From (1), (2) it follows that:*

$$\sum_{i=1}^n \frac{a_i}{\sqrt{\sum_{\substack{j=1 \\ j \neq i}}^n a_j^2}} \leq \frac{n}{2\sqrt{n-1}} \left( \frac{m}{M} + \frac{M}{m} \right)$$

**685. If  $a, b, c > 0$  then prove:**

$$\frac{b^4 + c^2}{a^3} + \frac{c^4 + a^2}{b^3} + \frac{a^4 + b^2}{c^3} \geq 6$$

*Proposed by Jalil Hajimir-Toronto-Canada*

**Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaijan**

Using AM-GM inequality, we have:  $b^4 + c^2 \geq 2b^2c, c^4 + a^2 \geq 2c^2a, a^4 + b^2 \geq 2a^2b$

And  $x + y + z \geq 3\sqrt[3]{xyz}$ . Hence,

$$\frac{b^4 + c^2}{a^3} + \frac{c^4 + a^2}{b^3} + \frac{a^4 + b^2}{c^3} \geq \frac{2b^2c}{a^3} + \frac{2c^2a}{b^3} + \frac{2a^2b}{c^3} \geq 3\sqrt[3]{\frac{2b^2c}{a^3} \cdot \frac{2c^2a}{b^3} \cdot \frac{2a^2b}{c^3}} = 6$$

**Solution 2 by Florică Anastase-Romania**

$$\begin{aligned}
 \frac{b^4 + c^2}{a^3} + \frac{c^4 + a^2}{b^3} + \frac{a^4 + b^2}{c^3} &= \left( \frac{b^4}{a^3} + \frac{c^4}{b^3} + \frac{a^4}{c^3} \right) + \left( \frac{c^2}{a^3} + \frac{a^2}{b^3} + \frac{b^2}{c^3} \right) \stackrel{\text{AGM}}{\geq} \\
 &\stackrel{\text{AGM}}{\geq} 3\sqrt[3]{abc} + \frac{3}{\sqrt[3]{abc}} \stackrel{\text{AGM}}{\geq} 6
 \end{aligned}$$

**Solution 3 by Khaled Abd Imouti-Damascus-Syria**

$$\frac{b^4 + c^2}{a^3} + \frac{c^4 + a^2}{b^3} + \frac{a^4 + b^2}{c^3} \stackrel{\text{AGM}}{\geq} 3\sqrt[3]{\frac{b^4 + c^2}{a^3} \cdot \frac{c^4 + a^2}{b^3} \cdot \frac{a^4 + b^2}{c^3}} =$$



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$$= 3 \frac{\sqrt[3]{(b^4 + c^2)(c^4 + a^2)(a^4 + b^2)}}{abc} \stackrel{AGM}{\geq} 3 \frac{\sqrt[3]{8a^2b \cdot b^2c \cdot c^2a}}{abc} = \\ = \frac{6\sqrt[3]{a^3b^3c^3}}{abc} = 6$$

**Solution 5 by Abdallah El Farissi-Bechar-Algerie**

$$\frac{b^4 + c^2}{a^3} + \frac{c^4 + a^2}{b^3} + \frac{a^4 + b^2}{c^3} = \frac{b^4}{a^3} + \frac{c^4}{b^3} + \frac{a^4}{c^3} + \frac{c^2}{a^3} + \frac{a^2}{b^3} + \frac{b^2}{c^3} \stackrel{AGM}{\geq} \\ \geq 6 \sqrt[6]{\frac{b^4}{a^3} \cdot \frac{c^4}{b^3} \cdot \frac{a^4}{c^3} \cdot \frac{c^2}{a^3} \cdot \frac{a^2}{b^3} \cdot \frac{b^2}{c^3}} = 6$$

**686. If  $x, y, z > 0, \sqrt{x} + \sqrt{y} + \sqrt{z} = 2, \lambda \geq 0$  then:**

$$\sum_{cyc} \frac{\sqrt{x^3} + \sqrt{y^3}}{x + \lambda\sqrt{xy} + y} \geq \frac{2}{\lambda + 2}$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Soumitra Mandal-Chandar Nagore-India**

Let  $a = \sqrt{x}, b = \sqrt{y}, c = \sqrt{z} \Rightarrow a + b + c = 2$ . So,

$$\sum_{cyc} \frac{\sqrt{x^3} + \sqrt{y^3}}{x + \lambda\sqrt{xy} + y} = \sum_{cyc} \frac{a^3 + b^3}{a^2 + \lambda ab + b^2} = \\ = \sum_{cyc} (a + b) \cdot \frac{a^2 + \lambda ab + b^2 - (\lambda + 1)ab}{a^2 + \lambda ab + b^2} = \\ = 2(a + b + c) - (\lambda + 1) \sum_{cyc} \frac{ab}{a^2 + \lambda ab + b^2} (a + b) \stackrel{(*)}{\geq}$$

$$\stackrel{(*)}{\geq} 4 - \frac{\lambda + 1}{\lambda + 2} \cdot 2(a + b + c) = 4 - 4 \cdot \frac{\lambda + 1}{\lambda + 2} = \frac{4}{\lambda + 1}$$

**Equality holds for  $a = b = c = \frac{2}{3} \Leftrightarrow x = y = z = \frac{4}{9}$ .**

(\*) : **Weighted AM-GM**

$$\frac{a^2 + \lambda ab + b^2}{\lambda + 1 + 1} \geq \sqrt[\lambda+2]{a^2 \cdot (ab)^\lambda \cdot b^2} \Rightarrow \frac{1}{\lambda + 2} \geq \frac{ab}{a^2 + \lambda ab + b^2}$$



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**Solution 2 by Florică Anastase-Romania**

Let  $a = \sqrt{x}$ ,  $b = \sqrt{y}$ ,  $c = \sqrt{z} \Rightarrow a + b + c = 2$ . So,

$$\begin{aligned}
\sum_{cyc} \frac{\sqrt{x^3} + \sqrt{y^3}}{x + \lambda\sqrt{xy} + y} &= \sum_{cyc} \frac{a^3 + b^3}{a^2 + \lambda ab + b^2} = \\
&= \sum_{cyc} (a + b) \cdot \frac{a^2 + \lambda ab + b^2 - (\lambda + 1)ab}{a^2 + \lambda ab + b^2} = \\
&= \sum_{cyc} (a + b) - (\lambda + 1) \sum_{cyc} \frac{ab(a + b)}{(a^2 + b^2) + \lambda ab} \stackrel{AGM}{\geq} \\
&= 4 - (\lambda + 1) \sum_{cyc} \frac{ab(a + b)}{(\lambda + 2)ab} = 4 - (\lambda + 1) \sum_{cyc} \frac{a + b}{(\lambda + 2)} = \\
&= 4 \left(1 - \frac{\lambda + 1}{\lambda + 2}\right) = \frac{4}{\lambda + 2}
\end{aligned}$$

Equality holds for  $a = b = c = \frac{2}{3} \Leftrightarrow x = y = z = \frac{4}{9}$ .

**Solution 3 by Abdul Hannan-Tezpur india**

Let  $a = \sqrt{x}$ ,  $b = \sqrt{y}$ ,  $c = \sqrt{z} \Rightarrow a + b + c = 2$

We will prove that:

$$\sum_{cyc} \frac{a^3 + b^3}{a^2 + \lambda ab + b^2} \geq \frac{2(a + b + c)}{\lambda + 2} \Leftrightarrow \sum_{cyc} \frac{a^3 + b^3}{a^2 + \lambda ab + b^2} \geq \sum_{cyc} \frac{a + b}{\lambda + 2}$$

It is enough to prove that:

$$\begin{aligned}
\frac{a^3 + b^3}{a^2 + \lambda ab + b^2} \geq \frac{a + b}{\lambda + 2} \Leftrightarrow (\lambda + 2)(a^2 - ab + b^2) &\geq a^2 + \lambda ab + b^2 \\
(\lambda + 1)(a - b)^2 \geq 0 \text{ which is true.}
\end{aligned}$$

Equality holds for  $a = b = c = \frac{2}{3} \Leftrightarrow x = y = z = \frac{4}{9}$ .

**Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand**

$$\sum_{cyc} \frac{\sqrt{x^3} + \sqrt{y^3}}{x + \lambda\sqrt{xy} + y} = \sum_{cyc} \frac{(\sqrt{x} + \sqrt{y})(\sqrt{x^2} - \sqrt{xy} + \sqrt{y^2})}{x + \lambda\sqrt{xy} + y} \geq$$



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$$\geq \sum_{cyc} \frac{\sqrt{x} + \sqrt{y}}{2(2 + \lambda)} = \frac{2}{\lambda + 2}$$

*Where,  $\frac{x - \sqrt{xy} + y}{x + y + \lambda\sqrt{xy}} \geq \frac{1}{2(\lambda+2)}$  (and analogs)*

**687. If  $x, y, z > 0$  and  $n \geq 2, k \leq 4, 2n + 3k = 16$  then:**

$$\frac{(x+y)(y+z)(z+x)}{xyz} \geq n + k \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$2n + 3k = 16 \Rightarrow k = \frac{16 - 2n}{3} \Rightarrow (i) \Leftrightarrow \frac{(x+y)(y+z)(z+x)}{xyz}$$

$$\geq n + \left( \frac{16 - 2n}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)$$

$$\Leftrightarrow \frac{(x+y)(y+z)(z+x)}{xyz} - \frac{16}{3} \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)$$

$$+ n \left( \left( \frac{2}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) - 1 \right) \stackrel{(ii)}{\geq} 0$$

$$\text{Now, } \frac{(x+y)(y+z)(z+x)}{xyz} - \frac{16}{3} \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right)$$

$$+ 2 \left( \left( \frac{2}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) - 1 \right) \geq 0$$

$$\Leftrightarrow \frac{(x+y)(y+z)(z+x)}{xyz} - 4 \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) \stackrel{(iii)}{\geq} 2$$

*Let  $y+z = a, z+x = b$  and  $x+y = c$  and since  $a+b > c, b+c > a$  and  $c+a > b$*

*$\therefore a, b, c$  are sides of a triangle whose*

*circumradius, inradius and semi-perimeter =  $R, r, s$  (say) and  $\because 2 \sum x$*

$$= a + b + c = 2s \Rightarrow \sum x = s$$



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$$\begin{aligned}
 & \because x = s - a, y = s - b \text{ and } z = s - c \therefore (iii) \Leftrightarrow \frac{4Rrs}{sr^2} - 4 \sum \frac{s-a}{a} \geq 2 \\
 & \Leftrightarrow \frac{4R}{r} + 10 - \frac{4s(s^2 + 4Rr + r^2)}{4Rrs} \geq 0 \\
 & \Leftrightarrow \frac{4R + 10r}{r} - \frac{s^2 + 4Rr + r^2}{Rr} \geq 0 \Leftrightarrow \frac{4R^2 + 10Rr - s^2 - 4Rr - r^2}{Rr} \geq 0 \Leftrightarrow s^2 \\
 & \leq 4R^2 + 6Rr - r^2 \\
 & \Leftrightarrow (s^2 - 4R^2 - 4Rr - 3r^2) - 2r(R - 2r) \leq 0 \rightarrow \text{true} \\
 & \because s^2 - 4R^2 - 4Rr - 3r^2 \stackrel{\text{Gerretsen}}{\geq} 0 \text{ and } -2r(R - 2r) \stackrel{\text{Euler}}{\geq} 0 \\
 & \Rightarrow (iii) \text{ is true} \\
 & \therefore \frac{(x+y)(y+z)(z+x)}{xyz} - \frac{16}{3} \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) \\
 & + 2 \left( \left( \frac{2}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) - 1 \right) \geq 0 \\
 & \Rightarrow \frac{(x+y)(y+z)(z+x)}{xyz} - \frac{16}{3} \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) \\
 & \geq -2 \left( \left( \frac{2}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) - 1 \right) \\
 & \stackrel{?}{\geq} -n \left( \left( \frac{2}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) - 1 \right) \\
 & \Leftrightarrow \left( \left( \frac{2}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) - 1 \right) (n-2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because n \geq 2 \\
 & \text{and } \left( \frac{2}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) - 1 \stackrel{\text{Nesbitt}}{\geq} \left( \frac{2}{3} \right) \left( \frac{3}{2} \right) - 1 = 0 \\
 & \therefore \frac{(x+y)(y+z)(z+x)}{xyz} - \frac{16}{3} \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) \\
 & \geq -n \left( \left( \frac{2}{3} \right) \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right) - 1 \right) \Rightarrow (ii) \Rightarrow (i) \text{ is true (Proved)}
 \end{aligned}$$

**Solution 2 by Michael Sterghiou-Greece**



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$$\frac{(x+y)(y+z)(z+x)}{xyz} \geq n + k \left( \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \right); \quad (1)$$

Let  $a = \frac{x}{y+z}$ ,  $b = \frac{y}{z+x}$ ,  $c = \frac{z}{x+y}$  and  $(p, q, r) = (\sum a, \sum ab, abc)$

$$p = \sum_{cyc} \left( \frac{x}{z} + \frac{z}{x} \right) \geq 6, r = \prod_{cyc} \frac{x+y}{z} = 2 + \sum_{cyc} \left( \frac{x}{z} + \frac{z}{x} \right) = 2 + p \geq 8$$

$$(1) \Rightarrow r \geq n + \frac{16 - 2k}{3} \cdot \sum_{cyc} \frac{1}{a} = n + \frac{16}{3} \cdot \frac{q}{r} - \frac{2k}{3} \cdot \frac{q}{r}$$

$$Or r + n \left( \frac{2}{3} \cdot \frac{q}{r} - 1 \right) - \frac{16}{3} \cdot \frac{q}{r} \geq 0; \quad (2)$$

$$But \frac{q}{r} = \sum_{cyc} \frac{1}{a} = \sum_{cyc} \frac{x}{y+z} \stackrel{\text{Nesbitt}}{\geq} \frac{3}{2}$$

Hence,  $\frac{2}{3} \cdot \frac{q}{r} \geq 1$  and (2) reduces to the stronger inequality:

$$r + 2 \cdot \frac{2}{3} \cdot \frac{q}{r} - 2 - \frac{16}{3} \cdot \frac{q}{r} \geq 0, n \geq 2$$

$$r^2 - 2r - 4q \geq 0; \quad (3)$$

From the 3-rd degree Schur's we have:

$$q \leq \frac{p^3 + 9r}{4p} = \frac{(r-2)^3 + 9r}{4(r-2)} \text{ and (2) becomes the stronger } r^2 - 2r - \frac{(r-2)^3 + 9r}{r-2} \geq 0$$

Therefore,

$$\frac{(r-8)(2r-1)}{r-2} \geq 0 \text{ which is true for } r \geq 8. \text{ Done!}$$

**688. If  $a, b, c > 0, a + b + c = 3$  then prove:**

$$\frac{1}{6} \left( \frac{a+b}{(ab)^2} + \frac{b+c}{(bc)^2} + \frac{c+a}{(ca)^2} \right) + \frac{a^3 + b^3 + c^3 + 2abc}{5} \geq 2$$

*Proposed by Pavlos Trifon-Greece*

*Solution by Abdul Hannan-Tezpur-India*

$$\begin{aligned} \frac{1}{6} \sum \frac{a+b}{a^2 b^2} &\stackrel{AGM}{\geq} \frac{1}{2} \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{a^4 b^4 c^4}} \stackrel{AGM}{\geq} \frac{1}{2} \sqrt[3]{\frac{8abc}{a^4 b^4 c^4}} = \frac{1}{2} \cdot \frac{2}{abc} = \frac{1}{abc} \stackrel{AGM}{\geq} \\ &\stackrel{AGM}{\geq} \frac{1}{\left(\frac{a+b+c}{3}\right)^3} = 1; \quad (1) \end{aligned}$$



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*On the other hand,*

$$\begin{aligned} \frac{a^3 + b^3 + c^3 + 2abc}{5} &= \frac{5abc + a^3 + b^3 + c^3 - 3abc}{5} = \\ &= 1 + \frac{1}{10}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 1; \quad (2) \end{aligned}$$

*From (1),(2) we obtain the desired result.*

*Equality holds when  $a = b = c = 1$ .*

**689. If  $m, n, p, q \in \mathbb{N}; m, n, p, q \geq 4$  then:**

$$4^n(4^n + 1) + 4^m(4^m + 1) + 4^p(4^p + 1) + 4^q(4^q + 1) \geq 4mnpq(mnpq + 1)$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Ravi Prakash-New Delhi-India**

Let  $f(x) = x^{\frac{1}{x}}, x \geq e$ ;  $\log f(x) = \frac{1}{x} \log x$

$\frac{f'(x)}{f(x)} = \frac{1}{x^2}(1 - \log x) < 0, \forall x \geq e$  the  $f$  – strictly decreasing on  $[e, \infty)$ .

Hence, If  $x \geq 4$ , then  $4^{\frac{1}{4}} \geq x^{\frac{1}{x}} \Rightarrow 4^x \geq x^4, \forall x \geq 4 \Rightarrow 4^{2x} + 4^x \geq x^8 + x^4$

Therefore,  $4^n(4^n + 1) + 4^m(4^m + 1) + 4^p(4^p + 1) + 4^q(4^q + 1) \geq$

$\geq n^8 + m^8 + p^8 + q^8 + n^4 + m^4 + p^4 + q^4 \geq$

$$\geq 4\sqrt[4]{n^8m^8p^8q^8} + 4\sqrt[4]{n^4m^4p^4q^4} = 4mnpq(mnpq + 1)$$

**690. Find  $x, y, z, t$  natural numbers such that:**

$$34 + \prod_{cyc} (2x+1)(2y+1)(2z+1) = 2 \prod_{cyc} (2x+1)$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Santos Martins Junior-Brussels-Belgium**

Let  $a = 2x + 2, b = 2y + 1, c = 2z + 1, d = 2t + 1$ , where  $a, b, c, d$  are all odds natural numbers. Let  $P = abcd$ , where  $P$  is odd.

Let's consider  $x, y, z, t \geq 1$  then we have that  $a, b, c, d \geq 3$ ; (1), implying  $P \geq 81$ ; (2)

Equation becomes as:  $34 + abc + bcd + cda + dab = 2abcd$



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$$\Leftrightarrow 34 + P \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) = \frac{2}{P} \Leftrightarrow 34 = P \left( 1 - \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \right)$$

$$\text{Let } M = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \text{ then } 34 = P(2 - M) \Leftrightarrow P = \frac{34}{2-M}$$

$$\text{Also, from (1): } \frac{1}{a} \leq \frac{1}{3}, \frac{1}{b} \leq \frac{1}{3}, \frac{1}{c} \leq \frac{1}{3}, \frac{1}{d} \leq \frac{1}{3}, \text{ implying } M \leq \frac{4}{3}; \quad (3)$$

**Now, using (2):**  $P = \frac{34}{2-M} \Leftrightarrow \frac{34}{2-M} = 81 \Leftrightarrow M \geq \frac{128}{81} > \frac{3}{2} > \frac{4}{3}$  which contradicts (3) then

**no solution in natural numbers for  $x, y, z, t \geq 1$ .**

**Assume two of  $x, y, z, t$  equals to 0, say  $x = y = 0$ ;**

**Keeping the same notation, the implies  $a = b = 1$ .**

**Then equation becomes:**  $34 + c + cd + cd + d = 2cd \Leftrightarrow 34 + c + d = 0$  no solution for natural numbers.

**Hence, we can at most have one of the  $x, y, z, t$  equals to 0, say  $x = 0 \Rightarrow a = 1$ .**

**Equation becomes:**  $34 + bc + bcd + cd + bd = 2bcd$

$$\Leftrightarrow 34 + bc + cd + db = bcd$$

$$\Leftrightarrow 34 + cd = b(cd - (c + d)) \Leftrightarrow \frac{34 + cd}{cd - (c + d)} = b; \quad (4)$$

$$\text{We put } b = 2y + 1, c = 2z + 1, d = 2t + 1$$

$$\text{Hence (4) becomes: } \frac{34 - (2z+1)(2t+1)}{(2z+1)(2t+1) - (2z+1+2t+1)} = 2y + 1$$

$$\Leftrightarrow \frac{18 + z + t}{4zt - 1} = y; \quad (5)$$

$$\text{Since } y \geq 1, \text{ we have } 18 + z + t = 4zt - 1 \Leftrightarrow 19 + z + t = 4zt; \quad (6)$$

**From (5): for  $z = 1$ :  $\frac{19+t}{4t-1} = y$  only for  $t = 2$  yielding  $y = 3$  and for  $t = 3$  yielding**

$$y = 2 \Rightarrow (y, z, t) = (3, 1, 2), (2, 1, 3)$$

**From (5): for  $z = 2$ :  $\frac{20+t}{8t-1} = y$ : only for  $t = 1$  yielding  $y = 3$  and for  $t = 3$  yielding**

$$y = 1 \Rightarrow (y, z, t) = (3, 2, 1), (1, 2, 3)$$

**From (5): for  $z = 3$ :  $\frac{21+t}{12t-1} = y$ : only for  $t = 1$  yielding  $y = 2$  and for  $t = 2$  yielding**

$$y = 1 \Rightarrow (y, z, t) = (2, 3, 1), (1, 3, 2)$$

**If  $z = 4$  from (6):  $23 + t > 16t$  only for  $t = 1$  but  $(z, t) = (4, 1)$  does not satisfy (5).**

**If  $z = 5$  from (6):  $24 + t \geq 20t$  only for  $t = 1$  but  $(z, t) = (5, 1)$  does not satisfy (5).**



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**If  $z = 6$  from (6):  $25 + t > 24t$  only for  $t = 1$  but  $(z, t) = (6, 1)$  does not satisfy (5)**

**If  $z = 7$  from (6):  $26 + t \geq 28t$  impossible. Solutions:**

$$(x, y, z, t) \in \{(0, 3, 1, 2), (0, 2, 1, 3), (0, 3, 2, 1), (0, 1, 2, 3), (0, 2, 3, 1), (0, 1, 3, 2)\}$$

**691. If  $x, y, z > 0$  then:**

$$\frac{(x+y)(y+z)(z+x)}{8xyz} \geq 1 + \left( \frac{1}{(x+y)^2} - \frac{1}{(x+y+2z)^2} \right) \cdot (x-y)^2$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution by Soumava Chakraborty-Kolkata-India**

Let  $y+z = a, z+x = b, x+y = c$  and then  $a+b > c, b+c > a$  and  $c+a > b$

$\Rightarrow a, b, c$  are sides of a triangle with

semiperimeter, circumradius, inradius  $= s, R, r$  respectively (say) and

$$\therefore 2 \sum x = \sum a = 2s \Rightarrow \sum x = s \therefore x = s - a, y = s - b$$

$$\text{and } z = s - c \therefore \frac{(x+y)(y+z)(z+x)}{8xyz} = \frac{abc}{8 \prod(s-a)} = \frac{4Rrs}{8r^2s}$$

$$= \frac{R}{2r} \text{ and } 1 + \left( \frac{1}{(x+y)^2} - \frac{1}{(x+y+2z)^2} \right) \cdot (x-y)^2$$

$$= 1 + \left( \frac{1}{c^2} - \frac{1}{(a+b)^2} \right) (a-b)^2 \therefore \frac{(x+y)(y+z)(z+x)}{8xyz}$$

$$\geq 1 + \left( \frac{1}{(x+y)^2} - \frac{1}{(x+y+2z)^2} \right) \cdot (x-y)^2$$

$$\Leftrightarrow \frac{R}{2r} - 1 \geq \left( \frac{1}{c^2} - \frac{1}{(a+b)^2} \right) (a-b)^2 = \left( \frac{(a+b)^2 - c^2}{c^2(a+b)^2} \right) (a-b)^2$$

$$\Leftrightarrow \frac{R}{2r} - 1 \stackrel{(a)}{\geq} \frac{4s(s-c)(a-b)^2}{c^2(a+b)^2}$$



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$$\text{Now, } r_a + r_b = s \left( \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} + \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} \right) = \frac{s \sin \left( \frac{A+B}{2} \right) \cos \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{C}{2}}{\left( \frac{s}{4R} \right)} = 4R \cos^2 \frac{C}{2}$$

$$\therefore r_a + r_b \stackrel{(i)}{=} 4R \cos^2 \frac{C}{2}$$

$$\text{Now, } (a+b)^2 \geq 32Rr \cos^2 \frac{C}{2} \stackrel{\text{by (i)}}{=} 8r(r_a + r_b) = 8r^2 s \left( \frac{1}{s-a} + \frac{1}{s-b} \right)$$

$$= 8(s-a)(s-b)(s-c) \frac{c}{(s-a)(s-b)} = 4c(a+b-c)$$

$$\Leftrightarrow (a+b)^2 + 4c^2 - 4c(a+b) \geq 0 \Leftrightarrow (a+b-2c)^2 \geq 0 \rightarrow \text{true} \therefore a+b$$

$$\geq 4\sqrt{2Rr} \cos \frac{C}{2} \Rightarrow 4R \cos \frac{C}{2} \cos \frac{A-B}{2} \geq 4\sqrt{2Rr} \cos \frac{C}{2}$$

$$\Rightarrow \cos \frac{A-B}{2} \geq \sqrt{\frac{2r}{R}} \Rightarrow \frac{1}{\cos^2 \frac{A-B}{2}} \stackrel{(ii)}{\leq} \frac{R}{2r}$$

$$\text{Now, } \frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} = \frac{\left(16R^2 \sin^2 \frac{A-B}{2} \sin^2 \frac{C}{2}\right) \left(16R^2 \cos^2 \frac{C}{2} \sin^2 \frac{A}{2} \sin^2 \frac{B}{2}\right)}{\left(16R^2 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}\right) \left(16R^2 \cos^2 \frac{A-B}{2} \cos^2 \frac{C}{2}\right)}$$

$$= \frac{1 - \cos^2 \frac{A-B}{2}}{\cos^2 \frac{A-B}{2}} = \frac{1}{\cos^2 \frac{A-B}{2}} - 1 \stackrel{\text{via (ii)}}{\leq} \frac{R}{2r} - 1$$

$$\Rightarrow \frac{R}{2r} - 1 \stackrel{(iii)}{\geq} \frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} \therefore (iii) \Rightarrow \text{in order to prove (a), it suffices to prove}$$

$$\therefore \frac{(a-b)^2(s-c)^2}{r^2(a+b)^2} \geq \frac{4s(s-c)(a-b)^2}{c^2(a+b)^2}$$

$$\Leftrightarrow \frac{(s-c)}{r^2} \geq \frac{4s}{c^2} \Leftrightarrow (s-c)c^2 \geq 4sr^2 = 4(s-a)(s-b)(s-c) \Leftrightarrow c^2$$

$$\geq (b+c-a)(c+a-b) = c^2 - (a-b)^2 \Leftrightarrow (a-b)^2 \geq 0$$

$$\rightarrow \text{true} \Rightarrow (a) \text{ is true} \therefore \forall x, y, z > 0, \frac{(x+y)(y+z)(z+x)}{8xyz}$$

$$\geq 1 + \left( \frac{1}{(x+y)^2} - \frac{1}{(x+y+2z)^2} \right) \cdot (x-y)^2 \text{ (Proved)}$$



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**692. If  $x, y, z > 0$  then:**

$$\sqrt[18]{\prod_{cyc} (1+x^2)} \cdot \sum_{cyc} \cos\left(\frac{\tan^{-1} x}{8}\right) \geq 3$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Tran Hong-Dong Thap-Vietnam**

For  $x, y, z \geq 0$ , let:  $x = \tan a; y = \tan b; z = \tan c \rightarrow a, b, c \in [0; \frac{\pi}{2})$

$$\rightarrow 0 < \cos a; \cos b; \cos c \leq 1; \cos \frac{a}{8}, \cos \frac{b}{8}, \cos \frac{c}{8} > 0$$

Now,

$$\begin{aligned} \sqrt[18]{\prod_{cyc} (1+x^2)} \cdot \sum_{cyc} \cos\left(\frac{\tan^{-1} x}{8}\right) \geq 3 &\Leftrightarrow \sqrt[18]{\prod_{cyc} \frac{1}{\cos^2 a}} \cdot \sum_{cyc} \cos \frac{a}{8} \geq 3; \\ \Leftrightarrow \sqrt[18]{\prod_{cyc} \frac{1}{\cos^2 a}} \cdot \cos \frac{a}{8} + \sqrt[18]{\prod_{cyc} \frac{1}{\cos^2 a}} \cdot \cos \frac{b}{8} + \sqrt[18]{\prod_{cyc} \frac{1}{\cos^2 a}} \cdot \cos \frac{c}{8} &\geq 3; \quad (1) \end{aligned}$$

Because:  $0 < \cos a; \cos b; \cos c \leq 1 \rightarrow 0 < \cos^2 a; \cos^2 b; \cos^2 c \leq 1$ , we have:

$$\begin{aligned} \sqrt[18]{\prod_{cyc} \frac{1}{\cos^2 a}} \cdot \cos \frac{a}{8} + \sqrt[18]{\prod_{cyc} \frac{1}{\cos^2 a}} \cdot \cos \frac{b}{8} + \sqrt[18]{\prod_{cyc} \frac{1}{\cos^2 a}} \cdot \cos \frac{c}{8} \\ \geq \sqrt[18]{\frac{1}{\cos^2 a}} \cdot \cos \frac{a}{8} + \sqrt[18]{\frac{1}{\cos^2 b}} \cdot \cos \frac{b}{8} + \sqrt[18]{\frac{1}{\cos^2 c}} \cdot \cos \frac{c}{8} \stackrel{(2)}{\geq} 3 \end{aligned}$$

We need to prove (2) is true. In fact, let:

$$\begin{aligned} \varphi(x) &= \sqrt[18]{\frac{1}{\cos^2 x}} \cdot \cos \frac{x}{8} = \sqrt[9]{\sec x} \cdot \cos \frac{x}{8}; \quad \forall x \in [0; \frac{\pi}{2}) \\ \rightarrow \varphi'(x) &= \frac{1}{72} \cdot \sqrt[9]{\sec x} \cdot \left(8 \cos \frac{x}{8} \tan x - 9 \sin \frac{x}{8}\right) \\ &= \frac{1}{72} \cdot \sqrt[9]{\sec x} \cdot \cos \frac{x}{8} \cdot \left(8 \tan x - 9 \tan \frac{x}{8}\right) \end{aligned}$$



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- Let:  $f(x) = \left(8 \tan x - 9 \tan \frac{x}{8}\right); \forall x \in \left[0; \frac{\pi}{2}\right] \rightarrow f'(x) = \frac{8}{\cos^2 x} - \frac{9}{8 \cos^2 \frac{x}{8}} =$

$$\frac{64 \cos^2 \frac{x}{8} - 9 \cos^2 x}{8 \cos^2 \frac{x}{8} \cos^2 x} > 0$$

$$(Since: \forall x \in \left[0; \frac{\pi}{2}\right] \rightarrow \cos \frac{x}{8} \geq \cos x > 0 \rightarrow$$

$$\cos^2 \frac{x}{8} \geq \cos^2 x \rightarrow 64 \cos^2 \frac{x}{8} - 9 \cos^2 x > 0)$$

*Therefore,*

$$f(x) \uparrow on \left[0; \frac{\pi}{2}\right] \rightarrow f(x) \geq f(0) = 0 \rightarrow \varphi'(x)$$

$$= \frac{1}{72} \cdot \sqrt[9]{\sec x} \cdot \left(8 \cos \frac{x}{8} \tan x - 9 \sin \frac{x}{8}\right) = \frac{1}{72} \cdot \sqrt[9]{\sec x} \cdot \cos \frac{x}{8} \cdot \left(8 \tan x - 9 \tan \frac{x}{8}\right) \geq 0$$

$$\rightarrow \varphi(x) \uparrow on \left[0; \frac{\pi}{2}\right] \rightarrow \varphi(x) \geq \varphi(0) = 1;$$

$\rightarrow \varphi(a) + \varphi(b) + \varphi(c) \geq 1 + 1 + 1 = 3 \rightarrow (2) is true \rightarrow (1) is true. Proved.$

**693. If  $x, y, z, t \in (0, 1)$  such that  $3\sqrt{3}(xyz + yzt + ztx + txy) = 4$  then**

**prove:**

$$\frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \geq 2$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Ravi Prakash-New Delhi-India**

$$Let f(t) = t(1-t^2), t \in (0, 1)$$

$$f'(t) = 1 - 3t^2 = 0 \Rightarrow t = \frac{1}{\sqrt{3}} \in (0, 1)$$

$$f''(t) = -6t$$

$$f''\left(\frac{1}{\sqrt{3}}\right) = -\frac{6}{\sqrt{3}} < 0$$

**Hence,  $f(t)$  –is maximum when  $t = \frac{1}{\sqrt{3}}$ , then  $f(t) \leq f\left(\frac{1}{\sqrt{3}}\right), \forall t \in (0, 1)$**

$$t - t^3 \leq \frac{2}{3\sqrt{3}}, \forall t \in (0, 1). Therefore,$$



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$$\begin{aligned}
 & \frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \geq \\
 & \geq \frac{3\sqrt{3}}{2} (xyz + yzt + ztx + txy) = 2
 \end{aligned}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $x, y, z, t \in (0, 1)$  such that  $3\sqrt{3}(xyz + yzt + ztx + txy) = 4$  we get:

$$\frac{4}{3\sqrt{3}} = xyz + yzt + ztx + txy \geq 4\sqrt[4]{(xyz)^3}$$

$$\text{Hence, } xyzt \leq \frac{1}{9}$$

$$\frac{4}{3\sqrt{3}} = xyz + yzt + ztx + txy \leq \frac{(x+y+z+t)^3}{4^2}$$

$$\text{Hence, } x^2 + y^2 + z^2 + t^2 \geq \frac{4}{3}$$

$$\begin{aligned}
 & \frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \geq \\
 & \geq \frac{(xyz + yz + zx + txy)^2}{4xyzt - (x^2 + y^2 + z^2 + t^2)xyzt} \geq 2
 \end{aligned}$$

$$\Leftrightarrow (xyz + yz + zx + txy)^2 + 2(x^2 + y^2 + z^2 + t^2)xyzt \geq 8xyzt$$

$$\text{Because } (xyz + yz + zx + txy)^2 = \frac{16}{27}$$

$$2(x^2 + y^2 + z^2 + t^2)xyzt \geq \frac{8}{3}xyzt \text{ and } xyzt \leq \frac{1}{9}$$

**694. If  $0 \leq a \leq b \leq c$  then prove:**

$$\left(\frac{a(a+c)}{b(b+c)}\right)^{\frac{c}{a+b}} \cdot \left(\frac{b(b+a)}{c(c+a)}\right)^{\frac{a}{b+c}} \cdot \left(\frac{c(c+b)}{a(a+b)}\right)^{\frac{b}{c+a}} \leq 1$$

*Proposed by Pavlos Trifon-Greece*

**Solution 1 by Abdul Hannan-Tezpur-India**

$$\text{Let } x = \frac{a}{b+c}, y = \frac{b}{c+a}, z = \frac{c}{a+b}$$

Now,  $0 < a \leq b \leq c \Rightarrow 0 < x \leq y \leq z$  and given inequality is equivalent to

$$\left(\frac{x}{y}\right)^z \left(\frac{y}{z}\right)^x \left(\frac{z}{x}\right)^y \leq 1 \Leftrightarrow x^{z-y} y^{y-x} \leq y^{z-x}$$



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*If  $x = z$ , then  $x = y = z$  and the inequality holds trivially.*

*Assume now that  $x < z$ .*

*By weighted AM-GM inequality, we obtain*

$$x^{\frac{z-y}{z-x}} \cdot z^{\frac{y-x}{z-x}} \leq \frac{x(z-y)}{z-x} + \frac{z(y-x)}{z-x} = y \text{ which proves the desired inequality}$$

**Solution 2 by proposer**

*For  $a = b$  or  $b = c$  is obviously.*

$$\text{For } 0 < a < b < c \Rightarrow \frac{a}{b+c} < \frac{b}{c+a} < \frac{c}{a+b}$$

*Let  $g(t) = \log t$  –concave on  $(0, \infty)$  hence, for  $0 < x < y < z$  it follows*

$$\log t \geq \frac{\log z - \log x}{z-x}(t-x) + \log x, \forall t \in [x, z]$$

*For  $t = y$  we get:*

$$\begin{aligned} \log y &> \frac{\log z - \log x}{z-x}(y-x) + \log x \\ \Rightarrow (z-x)\log y &> (y-x)\log z + (z-y)\log x; \quad (1) \end{aligned}$$

*By (1) and for  $x = \frac{a}{b+c}, y = \frac{b}{c+a}, z = \frac{c}{a+b} \Rightarrow$*

$$\left( \frac{a(a+c)}{b(b+c)} \right)^{\frac{c}{a+b}} \cdot \left( \frac{b(b+a)}{c(c+a)} \right)^{\frac{a}{b+c}} \cdot \left( \frac{c(c+b)}{a(a+b)} \right)^{\frac{b}{c+a}} \leq 1$$

**695. If  $a, b > 0$  then:**

$$\frac{a^9 + a^4b^4\sqrt{ab} + b^9}{(a^5 + a^2b^2\sqrt{ab} + b^5)^2} \geq \frac{a^4 + a^2b^2 + b^4}{(a^3 + ab\sqrt{ab} + b^3)(a^2 + ab + b^2)}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Rahim Shahbazov-Baku-Azerbaijan**

$$\frac{a^9 + a^4b^4\sqrt{ab} + b^9}{(a^5 + a^2b^2\sqrt{ab} + b^5)^2} \geq \frac{a^4 + a^2b^2 + b^4}{(a^3 + ab\sqrt{ab} + b^3)(a^2 + ab + b^2)}; \quad (1)$$

$$\Leftrightarrow (a^9 + a^4b^4\sqrt{ab} + b^9)(a^3 + ab\sqrt{ab} + b^3)(a^2 + ab + b^2) \geq$$



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$$\geq (a^4 + a^2b^2 + b^4)(a^5 + a^2b^2\sqrt{ab} + b^5)^2; \quad (2)$$

$$(a^9 + a^4b^4\sqrt{ab} + b^9)(a^3 + ab\sqrt{ab} + b^3) \stackrel{\text{Holder}}{\geq} (a^6 + a^3b^3 + b^6)^2$$

$$\text{Hence, } (a^6 + a^3b^3 + b^6)^2(a^2 + ab + b^2) =$$

$$(a^6 + a^3b^3 + b^6)(a^6 + a^3b^3 + b^6)(a^2 + ab + b^2) \stackrel{\text{Holder}}{\geq}$$

$$\stackrel{\text{Holder}}{\geq} (a^6 + a^3b^3 + b^6)(a^4 + a^2b^2 + b^4)^2 =$$

$$= (a^6 + a^3b^3 + b^6)(a^4 + a^2b^2 + b^4)(a^4 + a^2b^2 + b^4) \stackrel{\text{Holder}}{\geq}$$

$$\stackrel{\text{Holder}}{\geq} (a^4 + a^2b^2 + b^4)(a^5 + a^2b^2\sqrt{ab} + b^5)^2 \Rightarrow (2) \Rightarrow (1) \text{ is true.}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $a, b > 0$  we give  $a = x^2, b = y^2$ .

$$\frac{a^9 + a^4b^4\sqrt{ab} + b^9}{(a^5 + a^2b^2\sqrt{ab} + b^5)^2} \geq \frac{a^4 + a^2b^2 + b^4}{(a^3 + ab\sqrt{ab} + b^3)(a^2 + ab + b^2)}$$

$$\frac{x^{18} + (xy)^9 + y^{18}}{(x^{10} + (xy)^8 + y^{10})^2} \geq \frac{x^8 + (xy)^4 + y^8}{(x^6 + (xy)^3 + y^6)(x^4 + (xy)^2 + y^4)}$$

$$(x^{18} + (xy)^9 + y^{18})(x^6 + (xy)^3 + y^6)(x^4 + (xy)^2 + y^4) \geq$$

$$\geq (x^{10} + (xy)^8 + y^{10})^2(x^8 + (xy)^4 + y^8)$$

$$(x^{12} + (xy)^6 + y^{12})^2(x^4 + (xy)^2 + y^4) \geq (x^8 + (xy)^4 + y^8)(x^{10} + (xy)^5 + y^{10})^2$$

$$(x^8 + (xy)^4 + y^8)^2(x^{12} + (xy)^6 + y^{12}) \geq (x^8 + (xy)^4 + y^8)(x^{10} + (xy)^5 + y^{10})^2$$

$$(x^8 + (xy)^4 + y^8)(x^{10} + (xy)^5 + y^{10})^2 \geq (x^8 + (xy)^4 + y^8)(x^{10} + (xy)^5 + y^{10})^2$$

**696. If  $x, y > 0, xy = 1$  then:**

$$\frac{1}{x+3} + \frac{1}{y+3} \geq \frac{1}{x^2 + y^2}$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution 1 by Daniel Sitaru-Romania**



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$$\begin{aligned}
 a = x + \frac{1}{x} &\stackrel{AM-GM}{\geq} 2 \sqrt{x \cdot \frac{1}{x}} = 2 \rightarrow a - 2 \geq 0, x^2 + \frac{1}{x^2} = a^2 - 2 \\
 \frac{1}{x+3} + \frac{1}{y+3} &= \frac{1^2}{x+3} + \frac{1^2}{y+3} \stackrel{BERGSTROM}{\geq} \frac{(1+1)^2}{x+3+y+3} \geq \frac{1}{x^2+y^2} \\
 \Leftrightarrow 4(x^2+y^2) &\geq x+y+6 \Leftrightarrow 4\left(x^2+\frac{1}{x^2}\right) \geq x+\frac{1}{x}+6 \\
 \Leftrightarrow 4(a^2-2) &\geq a+6 \Leftrightarrow \\
 \Leftrightarrow 4a^2-a-14 &\geq 0 \Leftrightarrow (a-2)(4a+7) \geq 0
 \end{aligned}$$

*Solution 2 by Rustam Tahmazov-Azerbaijan*

$$\begin{aligned}
 \frac{1}{x+3} + \frac{1}{y+3} &\geq \frac{1}{x^2+y^2}; \quad (1) \\
 \Leftrightarrow \frac{x+y+6}{xy+3(x+y)+9} &\geq \frac{1}{x^2+y^2} \\
 (x+y)(x^2+y^2) + 6(x^2+y^2) &\geq 3(x+y) + 10 \\
 \text{But by AG-AM } x^2+y^2 &\geq 2xy = 2 \text{ and } 6(x^2+y^2) \geq 12 \\
 \Rightarrow (x+y)(x^2+y^2) + 2 &\geq 3(x+y) \\
 \Leftrightarrow x^3+x^2y+xy^2+y^3+2 &\geq 3(x+y) \\
 \Leftrightarrow x^3+y^3+2 &\geq 2(x+y)
 \end{aligned}$$

$$\begin{aligned}
 \text{But } x^3+y^3 &= (x+y)(x^2-xy+y^2); x+y \geq 2\sqrt{xy} = 2 \text{ hence,} \\
 2(x^2+y^2-1) + 2 &\geq 2(x+y) \\
 \Leftrightarrow 2(x^2+y^2) &\geq (x+y)^2 \\
 LHS \geq (x+y)^2 &\geq 2(x+y) \Rightarrow x+y \geq 2.
 \end{aligned}$$

*Solution 3 by George Florin Șerban-Romania*

$$\begin{aligned}
 \frac{1}{x+3} + \frac{1}{\frac{1}{x}+3} &\geq \frac{1}{x^2+\frac{1}{x^2}} \Rightarrow \frac{1}{x+3} + \frac{x}{3x+1} \geq \frac{x^2}{x^4+1} \\
 \Leftrightarrow \frac{x^2+6x+1}{3x^2+10x+3} &\geq \frac{x^2}{x^4+1} \\
 \Leftrightarrow x^6+6x^5+x^4+x^2+6x+1 &\geq 3x^4+10x^3+3x^2 \\
 \Leftrightarrow x^6+6x^5-2x^4-10x^3-2x^2+6x+1 &\geq 0
 \end{aligned}$$



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$$\begin{aligned} & \Leftrightarrow (x-1)(x^5 + 7x^4 + 5x^3 - 5x^2 - 7x - 1) \geq 0 \\ & \Leftrightarrow (x-1)^2(x^4 + 8x^3 + 13x^2 + 8x + 1) > 0, \forall x > 0 \text{ true.} \end{aligned}$$

**Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $x, y > 0$  and  $xy = 1$ , we give  $x = \frac{a}{b}, y = \frac{b}{a}$  hence,

$$\begin{aligned} & \frac{1}{x+3} + \frac{1}{y+3} \geq \frac{1}{x^2+y^2} \\ & \Leftrightarrow \frac{1}{\frac{a}{b}+3} + \frac{1}{\frac{b}{a}+3} \geq \frac{1}{\left(\frac{a}{b}\right)^2 + \left(\frac{b}{a}\right)^2} \\ & \text{Iff } \frac{b}{a+3b} + \frac{a}{b+3a} = \frac{b(b+3a)+a(a+3b)}{(a+3b)(b+3a)} \geq \frac{ab}{a^2+b^2} \geq \frac{(ab)^2}{a^4+b^4} \\ & \Leftrightarrow (a^2+b^2)(b^2+3ab+a^2+3ab) \geq ab(ab+3a^2+3b^2+9ab) \\ & \Leftrightarrow 3(a^3b+ab^3)+a^4+b^4 \geq 8a^2b^2 \text{ true.} \end{aligned}$$

**697. If  $a, b > 0$  then:**

$$(4ab)^{\sqrt{ab}} \cdot e^{30(a^2+b^2)} \leq (ae^{30a} + be^{30b})^{a+b}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by George Florin Șerban-Romania**

$$\begin{aligned} \frac{ae^{30a} + be^{30b}}{a+b} & \stackrel{\text{Holder}}{\geq} \sqrt[a+b]{ae^{30a} \cdot be^{30b}} = \sqrt[a+b]{e^{30(a^2+b^2)}} \\ & \Rightarrow ae^{30a} + be^{30b} \geq (a+b)^{a+b} \sqrt[e^{30(a^2+b^2)}]{} \\ & (ae^{30a} + be^{30b})^{a+b} \geq (a+b)^{a+b} \\ e^{30(a^2+b^2)} & = [(a=b)^2]^{\frac{a+b}{2}} e^{30(a^2+b^2)} \geq (4ab)^{\frac{a+b}{2}} \cdot e^{30(a^2+b^2)} \geq (4ab)^{\sqrt{ab}} \cdot e^{30(a^2+b^2)} \end{aligned}$$

**Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand**

For  $a, b > 0$  let  $a = x^2, b = y^2$ .

**Lemma.** For all  $m, n \in \mathbb{R}_+, m \geq n \Leftrightarrow m^m \geq n^n$

$$\begin{aligned} & \text{Consider } (4ab)^{\sqrt{ab}} \cdot e^{30(a^2+b^2)} \leq (ae^{30a} + be^{30b})^{a+b} \\ & (4x^2y^2)^{xy} \cdot e^{30(x^4+y^4)} \leq (x^2e^{30x^2} + y^2e^{30y^2})^{x^2+y^2} \end{aligned}$$



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$$(2xy)^{2xy} e^{30x^4} e^{30y^4} \leq (x^2 e^{30x^2} + y^2 e^{30y^2})^{x^2} (x^2 e^{30x^2} + y^2 e^{30y^2})^{y^2}$$

$$(2xy)^{2xy} \leq \frac{(x^2 e^{30x^2} + y^2 e^{30y^2})^{x^2} (x^2 e^{30x^2} + y^2 e^{30y^2})^{y^2}}{e^{30x^4} e^{30y^4}}$$

$$(2xy)^{2xy} \leq \left( \frac{x^2 + y^2}{\frac{x^2 e^{30x^2} + y^2 e^{30y^2}}{x^2 e^{30x^2} + y^2 e^{30y^2}}} \right)^{x^2 + y^2}$$

$$(2xy)^{2xy} \leq (x^2 + y^2)^{x^2 + y^2}$$

**698. If  $x, y, z > 0$  then:**

$$\frac{x}{\frac{y+z}{2} + \sqrt{2(y^2 + z^2)}} + \frac{y}{\frac{z+x}{2} + \sqrt{2(z^2 + x^2)}} + \frac{z}{\frac{x+y}{2} + \sqrt{2(x^2 + y^2)}} \geq 1$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution by Abdul Hannan-Tezpur-India**

$$\begin{aligned} \sum_{cyc} \frac{x}{\frac{y+z}{2} + \sqrt{2(y^2 + z^2)}} &= \sum_{cyc} \frac{x^2}{yx + zx + x\sqrt{2(y^2 + z^2)}} \stackrel{CBS}{\geq} \\ &= \frac{(\sum x)^2}{\sum xy + \sum \sqrt{2x}\sqrt{x(y^2 + z^2)}} \stackrel{CBS}{\geq} \frac{(\sum x)^2}{\sum xy + \sqrt{(\sum 2x)(\sum x(y^2 + z^2))}} \end{aligned}$$

*So, it is enough to prove that;*

$$\begin{aligned} \sum x^2 + \sum xy &\geq \sqrt{\left(\sum 2x\right)\left(\sum x(y^2 + z^2)\right)} \\ \left(\sum x^2 + \sum xy\right)^2 &\geq \left(\sum 2x\right)\left(\sum x(y^2 + z^2)\right) \\ \left(\sum x^2\right)^2 + \left(\sum xy\right)^2 + 2\left(\sum x\right)\left(\sum xy\right) &\geq \left(\sum 2x\right)\left(\sum x(y^2 + z^2)\right) \\ \left(\sum x^2\right)^2 + \left(\sum xy\right)^2 + 2\sum x^3(y+z) + 2\sum x^2yz &\geq \\ \geq 2\sum x^3(y+z) + 2\sum x^2(y+z)^2 & \\ \left(\sum x^2\right)^2 + \sum x^2y^2 + 2\sum x^2yz + 2\sum x^2yz &\geq 4\sum x^2y^2 + 4\sum x^2yz \end{aligned}$$



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$$(\sum x^2)^2 \geq 3 \sum x^2 y^2 \text{ which is true since}$$

$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

**699. If  $0 < x + y + z < \frac{\pi^2}{2}$  then:**

$$\frac{x \cos \sqrt{z} + y \cos \sqrt{x} + z \cos \sqrt{y}}{\cos \sqrt{\frac{xy + yz + zx}{x+y+z}}} \geq x + y + z$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Adrian Popa-Romania**

$x, y, z > 0$  and  $0 < x + y + z < \frac{\pi^2}{2}$  then  $\sqrt{x}, \sqrt{y}, \sqrt{z} \in \left(0, \frac{\pi}{2}\right)$

*Let be the function  $f(x) = \cos \sqrt{x}$*

$$f'(x) = -\frac{\sin \sqrt{x}}{2\sqrt{x}}; f''(x) = \frac{(\tan \sqrt{x} - \sqrt{x}) \cos \sqrt{x}}{4x} > 0, \forall \sqrt{x} \in \left(0, \frac{\pi}{2}\right), \tan \sqrt{x} - \sqrt{x} \geq 0$$

*Applying Jensen Inequality, we get:*

$$\frac{x \cos \sqrt{z} + y \cos \sqrt{x} + z \cos \sqrt{y}}{x + y + z} \geq \cos \sqrt{\frac{xy + yz + zx}{x+y+z}}$$

$$\text{Therefore, } \frac{x \cos \sqrt{z} + y \cos \sqrt{x} + z \cos \sqrt{y}}{\cos \sqrt{\frac{xy + yz + zx}{x+y+z}}} \geq x + y + z$$

**700. If  $a, b, c > 0$  then:**

$$\left(1 + \frac{e}{e^{a+b+c}}\right) \prod_{cyc} (1 + e^a)^a \geq (1 + \sqrt[4]{e}) \left(1 + \frac{e}{e^{a+b+c}}\right)^{a+b+c}$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Khaled Abd Imouti-Damascus-Syria**

$$\left(1 + \frac{e}{e^{a+b+c}}\right) \prod_{cyc} (1 + e^a)^a \geq (1 + \sqrt[4]{e}) \left(1 + \frac{e}{e^{a+b+c}}\right)^{a+b+c}$$



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$$\Leftrightarrow \left(1 + \frac{e}{e^{a+b+c}}\right)^{1-(a+b+c)} \prod_{cyc} (1 + e^a)^a \geq (1 + \sqrt[4]{e})$$

$$\Leftrightarrow (1 + e^{1-(a+b+c)})^{1-(a+b+c)} \prod_{cyc} (1 + e^a)^a \geq (1 + \sqrt[4]{e})$$

*Let be the function:  $f(x) = \log(1 + e^x)^x, x > 0, f'(x) = \log(1 + e^x) + \frac{xe^x}{1+e^x}$*

$$f''(x) = \frac{e^x(1+x+e^x+e^{2x})}{(1+e^x)^2} > 0 \Rightarrow f - \text{convex function, hence,}$$

$$f(1 - (a + b + c)) + f(a) + f(b) + f(c) \geq 4f\left(\frac{1 - a - b - c + a + b + c}{4}\right)$$

$$\log(1 + e^{1-(a+b+c)})^{1-(a+b+c)} + \sum_{cyc} \log(1 + e^a)^a \geq 4\log\left(1 + e^{\frac{1}{4}}\right)^{\frac{1}{4}}$$

$$\log\left[\left(1 + \frac{e}{e^{a+b+c}}\right)^{1-(a+b+c)} \prod_{cyc} (1 + e^a)^a\right] \geq \log(1 + \sqrt[4]{e})$$

$$\left(1 + \frac{e}{e^{a+b+c}}\right) \prod_{cyc} (1 + e^a)^a \geq (1 + \sqrt[4]{e}) \left(1 + \frac{e}{e^{a+b+c}}\right)^{a+b+c}$$



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*It's nice to be important but more important it's to be nice.*

*At this paper works a TEAM.*

*This is RMM TEAM.*

*To be continued!*

*Daniel Sitaru*