

RMM - Calculus Marathon 1101 - 1200

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DANIEL SITARU

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Proposed by

Daniel Sitaru – Romania, Srinivasa Raghava-AIRMC-India

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Radu Diaconu-Romania, Marin Chirciu-Romania, Abdul Mukhtar-

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Daniel Sitaru-Romania, Kamel Benaicha-Algiers-Algerie

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1101. Prove that:

$$\prod_{k=0}^{\infty} \exp\left(\frac{2F_{2k+1}}{(2k+1)5^k}\right) = 11$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Benaicha-Algiers-Algerie

$$P = \prod_{k=0}^{\infty} \exp\left(\frac{2F_{2k+1}}{(2k+1)5^k}\right) = \exp\left(\sum_{k=0}^{\infty} \frac{2F_{2k+1}}{(2k+1)5^k}\right)$$

$$F_{2k+1} = \frac{1}{\sqrt{5}} \left(\varphi^{2k+1} + \frac{1}{\varphi^{k+1}} \right) \Rightarrow P = \exp\left(2 \sum_{k=0}^{\infty} \frac{\left(\frac{\varphi}{\sqrt{5}}\right)^{2k+1} + \left(\frac{1}{\sqrt{5}\varphi}\right)^{2k+1}}{2k+1} \right)$$

Put:

$$f(x) = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \frac{x^{2k+1} + \frac{1}{x^{2k+1}}}{(2k+1)5^k} = \tanh^{-1}\left(\frac{x}{\sqrt{5}}\right) + \tanh^{-1}\left(\frac{1}{\sqrt{5}x}\right)$$

$$x = \varphi \Rightarrow f(\varphi) = \sum_{k=0}^{\infty} \frac{\left(\frac{\varphi}{\sqrt{5}}\right)^{2k+1} + \left(\frac{1}{\sqrt{5}\varphi}\right)^{2k+1}}{2k+1}$$

$$f(\varphi) = \frac{1}{2} \left(\log\left(\frac{1 + \frac{\varphi}{\sqrt{5}}}{1 - \frac{\varphi}{\sqrt{5}}}\right) + \log\left(\frac{1 + \frac{1}{\sqrt{5}\varphi}}{1 - \frac{1}{\sqrt{5}\varphi}}\right) \right) = \frac{1}{2} \log\left(\frac{\sqrt{5} + \varphi}{\sqrt{5} - \varphi} \cdot \frac{\sqrt{5}\varphi + 1}{\sqrt{5}\varphi - 1}\right) =$$

$$= \frac{1}{2} \log\left(\frac{6\varphi + \sqrt{5}\varphi^2 + \sqrt{5}}{6\varphi - \sqrt{5}\varphi - \sqrt{5}}\right) = \frac{1}{2} \log\left(\frac{6\varphi + \sqrt{5}(\varphi^2 + 1)}{6\varphi - \sqrt{5}(\varphi^2 + 1)}\right) =$$

$$= \frac{1}{2} \log\left(\frac{6\varphi + \sqrt{5}(2 + \varphi)}{6\varphi - \sqrt{5}(2 + \varphi)}\right) = \frac{1}{2} \log\left(\frac{6 + 6\sqrt{5} + 4\sqrt{5} + \sqrt{5} + 5}{6 + 6\sqrt{5} - 4\sqrt{5} - \sqrt{5} - 5}\right) =$$

$$= \frac{1}{2} \log\left(\frac{11 + 11\sqrt{5}}{1 + \sqrt{5}}\right) = \frac{1}{2} \log 11$$

$$p = \exp(2f(\varphi)) = \exp(\log 11) = 11$$



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$$\prod_{k=0}^{\infty} \exp\left(\frac{2F_{2k+1}}{(2k+1)5^k}\right) = 11$$

1102. Prove the relationship:

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sin(2x) \cos^3(\log(\tan x)) dx = 1 + \sum_{k=1}^{\infty} \frac{6(-1)^k(4k^2 + 3)}{16k^4 + 40k^2 + 9} = \\ & = \sum_{k=0}^{\infty} \frac{3\pi(1 + e^{2k\pi+\pi})}{4e^{\frac{3}{2}(2k\pi+\pi)}} = \frac{3}{8}\pi \sum_{k=1}^{\infty} \frac{8(-1)^k(4k^2 + 3)}{\pi(4k^2 + 1)(4k^2 + 9)} = \frac{3}{8}\pi \left(\operatorname{csch}\left(\frac{\pi}{2}\right) + \operatorname{csch}\left(\frac{3\pi}{2}\right) \right) \end{aligned}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} \Omega &= \int_0^{\frac{\pi}{2}} \sin(2x) \cos^3(\log(\tan x)) dx \stackrel{t=\tan x}{=} \int_0^{\infty} \frac{2t \cos^3(\log t)}{(1+t^2)^2} dt \stackrel{t \rightarrow t^2}{=} \\ &= \int_0^{\infty} \frac{\cos^3\left(\frac{1}{2}\log t\right)}{(1+t)^2} dt = \int_0^1 \frac{\cos^3\left(\frac{1}{2}\log t\right)}{(1+t)^2} dt + \underbrace{\int_1^{\infty} \frac{\cos^3\left(\frac{1}{2}\log t\right)}{(1+t)^2} dt}_{t=\frac{1}{z}} = \\ &= 2 \int_0^1 \frac{\cos^3\left(\frac{1}{2}\log t\right)}{(1+t)^2} dt \stackrel{IBP}{=} -2 \left[\left(\frac{1}{t+1} - 1 \right) \cos^3\left(\frac{1}{2}\log t\right) \right]_0^1 - \\ &\quad - 3 \int_0^1 \frac{\sin\left(\frac{1}{2}\log t\right) \cos^2\left(\frac{1}{2}\log t\right)}{t} \left(\frac{1}{1+t} - 1 \right) dt = \\ &= 1 + 3 \int_0^1 \frac{\sin\left(\frac{1}{2}\log t\right) \frac{1 + \cos(\log t)}{2}}{1+t} dt \stackrel{\sin a \cos(2a) = \frac{1}{2}(\sin(3a) - \sin a)}{=} \\ &= 1 + \frac{3}{4} \int_0^1 \frac{\sin\left(\frac{3}{2}\log t\right) + \sin\left(\frac{1}{2}\log t\right)}{1+t} dt \stackrel{u=-\log t}{=} \end{aligned}$$



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$$\begin{aligned}
 &= 1 - \frac{3}{4} \int_0^\infty \frac{\sin\left(\frac{3u}{2}\right) + \sin\left(\frac{u}{2}\right)}{1 + e^{-u}} \cdot e^{-u} du = \\
 &= 1 - \frac{3}{4} \sum_{n=0}^{\infty} (-1)^n \int_0^\infty \left(\sin\left(\frac{3u}{2}\right) + \sin\left(\frac{u}{2}\right) \right) e^{-(n+1)u} du = \\
 &\quad \left(\because \int_0^\infty \sin(ax) e^{-bx} dx = \mathcal{L}(\sin(ax)) = \frac{a}{b^2 + a^2} \right) \\
 &= 1 - \frac{3}{4} \sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{4(n+1)^2 + 9} + \frac{2}{4(n+1)^2 + 1} \right) = \\
 &= 1 + \frac{3}{4} \sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{4n^2 + 9} + \frac{2}{4n^2 + 1} \right) = 1 + \frac{3}{4} \sum_{n=1}^{\infty} \frac{(-1)^n(32n^2 + 24)}{(4n^2 + 1)(4n^2 + 9)} = \\
 &= 1 + 6 \sum_{n=1}^{\infty} \frac{(-1)^n(4n^2 + 3)}{(4n^2 + 1)(4n^2 + 9)} = 1 + 6 \sum_{n=1}^{\infty} \frac{(-1)^n(4n^2 + 3)}{16n^4 + 40n^2 + 9} \stackrel{(1)}{=} \\
 &= 1 + 3 \sum_{n=1}^{\infty} \frac{(-1)^n(4n^2 + 3)}{(4n^2 + 1)(4n^2 + 9)} + 3 \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{-n}(4(-n)^2 + 3)}{(4(-n)^2 + 1)(4(-n)^2 + 9)}}_{k=-n} = \\
 &= 3 \sum_{n=1}^{\infty} \frac{(-1)^n(4n^2 + 3)}{(4n^2 + 1)(4n^2 + 9)} + 3 \sum_{k=-\infty}^{-1} \frac{(-1)^k(4k^2 + 3)}{(4k^2 + 1)(4k^2 + 9)} = \\
 &= 3 \sum_{n=-\infty}^{\infty} \frac{(-1)^n(4n^2 + 3)}{(4n^2 + 1)(4n^2 + 9)} = \frac{3\pi}{8} \sum_{n=-\infty}^{\infty} \frac{8(-1)^n(4n^2 + 3)}{\pi(4n^2 + 1)(4n^2 + 9)} \stackrel{(2)}{=} \\
 &\Omega = 1 + 6 \sum_{n=1}^{\infty} \frac{(-1)^n(4n^2 + 3)}{(4n^2 + 1)(4n^2 + 9)} \\
 &\frac{4x + 3}{(4x + 1)(4x + 9)} = \frac{A}{4x + 1} + \frac{B}{4x + 9} \dots (E) \\
 &(E) \cdot x, x \rightarrow \infty \Rightarrow A + B = 1 \\
 &(E) \cdot (4x + 1), x = -\frac{1}{4} \Rightarrow A = \frac{1}{4}; B = \frac{3}{4}
 \end{aligned}$$

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$$\Omega = 1 + \frac{3}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 + 1} + \frac{9}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 + 9} = 1 + \frac{3}{8} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \left(\frac{1}{2}\right)^2} + \frac{9}{8} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \left(\frac{3}{2}\right)^2}$$

$$\left(\therefore \sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n^2 + a^2} = 2 \sum_{n=1}^{\infty} \frac{1}{4n^2 + a^2} \right)$$

$$\left(\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2 + \left(\frac{a}{2}\right)^2} - \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \right)$$

We know that: $\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\alpha \pi \cot(\alpha \pi) - 1}{2\alpha^2}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2 + \left(\frac{1}{4}\right)^2} - \sum_{n=1}^{\infty} \frac{1}{n^2 + \left(\frac{1}{2}\right)^2} = \frac{\frac{\pi}{4} \coth\left(\frac{\pi}{4}\right) - 1}{\frac{1}{4}} - \pi \coth\left(\frac{\pi}{2}\right) + 2$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \left(\frac{3}{2}\right)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2 + \left(\frac{3}{4}\right)^2} - \sum_{n=1}^{\infty} \frac{1}{n^2 + \left(\frac{3}{2}\right)^2} =$$

$$= \frac{\frac{3\pi}{4} \coth\left(\frac{3\pi}{4}\right) - 1}{\frac{9}{4}} - \frac{1}{3} \pi \coth\left(\frac{3\pi}{2}\right) + \frac{2}{9}$$

$$\Omega = 1 + \frac{1}{8} \left(3\pi \coth\left(\frac{\pi}{4}\right) - 12 - 3\pi \coth\left(\frac{\pi}{2}\right) + 6 + 3\pi \coth\left(\frac{3\pi}{4}\right) - 4 - 3\pi \coth\left(\frac{3\pi}{2}\right) + 2 \right) =$$

$$= 1 + \frac{1}{8} \left(3\pi \left(\coth\left(\frac{\pi}{4}\right) + \coth\left(\frac{3\pi}{4}\right) \right) - 3\pi \left(\coth\left(\frac{\pi}{2}\right) + \coth\left(\frac{3\pi}{2}\right) \right) - 8 \right)$$

$$\coth\left(\frac{x}{2}\right) = \frac{\cosh\left(\frac{x}{2}\right)}{\sinh\left(\frac{x}{2}\right)} = \frac{1 + \cosh(x)}{\sinh(x)} = \frac{1}{\sinh(x)} + \coth(x)$$

$$\coth\left(\frac{\pi}{4}\right) + \coth\left(\frac{3\pi}{4}\right) = \frac{1}{\sinh\left(\frac{\pi}{2}\right)} + \coth\left(\frac{\pi}{2}\right) + \frac{1}{\sinh\left(\frac{3\pi}{2}\right)} + \coth\left(\frac{3\pi}{2}\right)$$

$$So, \Omega = \frac{3\pi}{8} \left(\frac{1}{\sinh\left(\frac{\pi}{2}\right)} + \frac{1}{\sinh\left(\frac{3\pi}{2}\right)} \right) = \frac{3\pi}{8} \left(\operatorname{csch}\left(\frac{\pi}{2}\right) + \operatorname{csch}\left(\frac{3\pi}{2}\right) \right)$$

$$\begin{aligned}
 &= \frac{3\pi}{4} \left(\frac{1}{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}} + \frac{1}{e^{\frac{3\pi}{2}} - e^{-\frac{3\pi}{2}}} \right) = \frac{3\pi}{4} \left(\frac{e^{-\frac{\pi}{2}}}{1 - e^{-\pi}} + \frac{e^{-\frac{3\pi}{2}}}{1 - e^{-3\pi}} \right) = \\
 &= \frac{3\pi}{4} \sum_{n=0}^{\infty} \left(e^{-\frac{\pi}{2}} \cdot e^{-n\pi} + e^{-\frac{3\pi}{2}} \cdot e^{-3n\pi} \right) = \frac{3\pi}{4} \sum_{n=0}^{\infty} \left(e^{-\frac{\pi}{2}(2n+1)} + e^{-\frac{3\pi}{2}(2n+1)} \right) = \\
 &= \frac{3\pi}{4} \sum_{n=0}^{\infty} \left(\frac{1}{e^{\frac{\pi}{2}(2n+1)}} + \frac{1}{e^{\frac{3\pi}{2}(2n+1)}} \right) = \frac{3\pi}{4} \sum_{n=0}^{\infty} \frac{1 + e^{\frac{3\pi}{2}-\frac{\pi}{2}(2n+1)}}{e^{\frac{3\pi}{2}(2n+1)}} = \frac{3\pi}{4} \sum_{n=0}^{\infty} \frac{1 + e^{2n\pi+\pi}}{e^{\frac{3}{2}(2n\pi+\pi)}}
 \end{aligned}$$

1103. Find:

$$\Omega = \int_0^{\infty} (e^x \log(1 - e^{-x}) + 1)^2 dx$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned}
 \Omega &= \int_0^{\infty} (e^x \log(1 - e^{-x}) + 1)^2 dx = \int_0^{\infty} \frac{(\log(1 - e^{-x}) + e^{-x})^2}{e^{-3x}} \cdot e^{-x} dx \stackrel{t=e^{-x}}{=} \\
 &= \int_0^1 \frac{(\log(1-t) + t)^2}{t^3} dt \\
 f(t) &= \int \frac{(\log(1-t) + t)^2}{t^3} dt = \int \frac{\log^2(1-t)}{t^3} dt + 2 \int \frac{\log(1-t)}{t^2} dt + \int \frac{dt}{t} \stackrel{IBP}{=} \\
 &= -\frac{\log^2(1-t)}{2t^2} - \int \frac{\log(1-t)}{t^2(1-t)} dt + 2 \int \frac{\log(1-t)}{t^2} dt + \log t = \\
 &= -\frac{\log^2(1-t)}{2t^2} - \int \frac{\log(1-t)}{t^2} dt - \int \frac{\log(1-t)}{t(1-t)} dt + \log t = \\
 &= -\frac{\log^2(1-t)}{2t^2} - \frac{\log(1-t)}{t} - \int \frac{dt}{t(1-t)} - \int \frac{\log(1-t)}{t(1-t)} dt + \log t = \\
 &= -\frac{\log^2(1-t)}{2t^2} - \frac{\log(1-t)}{t} + \log(1-t) + Li_2(t) + \frac{1}{2} \log^2(1-t) + C; C \in \mathbb{R}
 \end{aligned}$$



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$$f(t) = \frac{\log^2(1-t)}{2} \left(\frac{(t-1)(t+1)}{t^2} \right) + \log(1-t) \left(\frac{t-1}{t} \right) + Li_2(t) + C$$

$$\therefore \Omega = \lim_{t \rightarrow 1^-}^A f(t) = \lim_{t \rightarrow 1^+}^B f(t) = A - B$$

$$\lim_{t \rightarrow 1^-} (1-t) \log^2(1-t) = \lim_{x \rightarrow 0^+} x \log^2 x = 4 \lim_{x \rightarrow 0^+} (\sqrt{x} \log \sqrt{x})^2 = 0 =$$

$$= \lim_{t \rightarrow 1^-} (1-t) \log(1-t)$$

$$\lim_{t \rightarrow 0^+} \frac{\log(1-t)}{t} = -1 \Rightarrow$$

$$B = \lim_{t \rightarrow 0^+} f(t) = \lim_{t \rightarrow 0^+} \left[\frac{\log^2(1-t)}{2} \left(\frac{(t-1)(t+1)}{t^2} \right) + \log(1-t) \left(\frac{t-1}{t} \right) + Li_2(t) \right] =$$

$$= -\frac{1}{2} + 1 + C = \frac{1}{2} + C$$

$$A = \lim_{t \rightarrow 1^-} f(t) = \frac{\pi^2}{6} + C$$

$$\Omega = \int_0^\infty (e^x \log(1-e^{-x}) + 1)^2 dx = \frac{\pi^2}{6} - \frac{1}{2}$$

1104. Prove that:

$$\int_0^\infty \frac{e^{-\pi \sqrt[3]{1+\sqrt[3]{z}}}}{\sqrt[3]{z}} dz = \frac{27e^{-\pi}}{\pi^6} (\pi(\pi(\pi(6+\pi) + 20) + 40) + 40)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Benaicha-Algiers-Algerie

$$\Omega = \int_0^\infty \frac{e^{-\pi \sqrt[3]{1+\sqrt[3]{z}}}}{\sqrt[3]{z}} dz$$

$$\text{Put: } t = \sqrt[3]{1 + \sqrt[3]{z}} \Rightarrow z = (t^3 - 1)^3 \Rightarrow dz = 9t^2(t^3 - 1)^2 dt$$

$$\Omega = 9 \int_1^\infty t^2(t^3 - 1)e^{-\pi t} dt = 9 \int_1^\infty (t^5 - t^2)e^{-\pi t} dt$$



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$$I_n = \int_1^\infty t^n e^{-\pi t} dt \stackrel{IBP}{=} \frac{e^{-\pi}}{\pi} + \frac{n}{\pi} \int_1^\infty t^{n-1} e^{-\pi t} dt = \frac{e^{-\pi}}{\pi} + \frac{n}{\pi} I_{n-1}; I_0 = \int_1^\infty e^{-\pi t} dt = \frac{e^{-\pi}}{\pi}$$

$$\Omega = 9(I_5 - I_2)$$

$$I_5 = \left(\frac{1}{\pi} + \frac{5}{\pi^2} + \frac{20}{\pi^3} + \frac{60}{\pi^4} + \frac{120}{\pi^5} + \frac{120}{\pi^6} \right) e^{-\pi}$$

$$I_2 = \left(\frac{1}{\pi} + \frac{2}{\pi^2} + \frac{2}{\pi^3} \right) e^{-\pi}$$

$$\begin{aligned} So: \Omega &= 9 \left(\frac{3}{\pi^2} + \frac{18}{\pi^3} + \frac{60}{\pi^4} + \frac{120}{\pi^5} + \frac{120}{\pi^6} \right) e^{-\pi} = \frac{27e^{-\pi}}{\pi^6} (40 + 40\pi + 20\pi^2 + 6\pi^3 + \pi^4) \\ &= \frac{27e^{-\pi}}{\pi^6} (\pi(\pi^3 + 6\pi^2 + 20\pi + 40) + 40) = \\ &= \frac{27e^{-\pi}}{\pi^6} (\pi(\pi(\pi^2 + 6\pi + 20) + 40) + 40) = \\ &= \frac{27e^{-\pi}}{\pi^6} (\pi(\pi(\pi(6 + \pi) + 20) + 40) + 40) \\ \int_0^\infty \frac{e^{-\pi \sqrt[3]{1+\sqrt[3]{z}}}}{\sqrt[3]{z}} dz &= \frac{27e^{-\pi}}{\pi^6} (\pi(\pi(\pi(6 + \pi) + 20) + 40) + 40) \end{aligned}$$

1105. Solve:

$$y(x) = 1 + 2 \int_0^x e^{-(x-t)} y(t) dt$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by Yen Tung Chung-Taichung-Taiwan

$$\begin{aligned} y(x) &= 1 + 2 \int_0^x e^{-(x-t)} y(t) dt = 1 + 2e^{-x} \int_0^x e^t y(t) dt \\ \frac{dy}{dx} &= \frac{d}{dx} \left(1 + 2e^{-x} \int_0^x e^t y(t) dt \right) = -2e^{-x} \int_0^x e^t y(t) dt + 2e^{-x} e^x y(x) \\ \Rightarrow \frac{dy}{dx} &= 1 - y(x) + 2y(x) \Rightarrow \frac{dy}{dx} = 1 + y \Rightarrow \frac{1}{1+y} dy = dx \end{aligned}$$



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$$\int \frac{1}{1+y} dy = \int dx + k \Rightarrow \log|1+y| = x + k \Rightarrow y = e^x c - 1$$

$$\text{Since } y(0) = 1 + 2 \int_0^0 e^{-(0-t)} y(t) dt = 1$$

We have: $1 = c - 1 \Rightarrow c = 2 \Rightarrow y = 2e^x - 1$

Solution 2 by Nassim Nicholas Taleb-New York-USA

Taking Laplace transforms on both sides

$$\mathcal{L}_x[y](s) = \mathcal{L}_x \left[2 \int_0^x e^{-(x-t)} y(t) dt + 1 \right] (s) = \frac{2(\mathcal{L}_x[y(x)](s))}{s+1} + \frac{1}{s}$$

$$\text{Hence: } \mathcal{L}_x[y(x)](s) = \frac{1+s}{(-1+s)s}$$

Inverting the Laplace Transform

$$y(x) = \mathcal{L}_s^{-1} \left[\frac{s+1}{(s-1)s} \right] (x) = 2e^x - 1$$

1106. Find without any software:

$$\int_0^{\frac{\pi}{2}} \frac{4 \sin x + 2\pi}{\sin x + \cos x + \pi} dx$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru – Romania

$$\begin{aligned} \Omega &= \int_0^{\frac{\pi}{2}} \frac{4 \sin x + 2\pi}{\sin x + \cos x + \pi} dx \stackrel{y=\frac{\pi}{2}-x}{=} \int_{\frac{\pi}{2}}^0 \frac{4 \sin \left(\frac{\pi}{2}-y\right) + 2\pi}{\sin \left(\frac{\pi}{2}-y\right) + \cos \left(\frac{\pi}{2}-y\right) + \pi} (-dy) = \\ &= \int_0^{\frac{\pi}{2}} \frac{4 \sin \left(\frac{\pi}{2}-y\right) + 2\pi}{\sin \left(\frac{\pi}{2}-y\right) + \cos \left(\frac{\pi}{2}-y\right) + \pi} dy = \int_0^{\frac{\pi}{2}} \frac{4 \cos y + 2\pi}{\cos y + \sin y + \pi} dy = \\ &= \int_0^{\frac{\pi}{2}} \frac{4 \cos x + 2\pi}{\sin x + \cos x + \pi} dx \end{aligned}$$



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$$\Omega + \Omega = \int_0^{\frac{\pi}{2}} \frac{4\sin x + 2\pi}{\sin x + \cos x + \pi} dx + \int_0^{\frac{\pi}{2}} \frac{4\cos x + 2\pi}{\sin x + \cos x + \pi} dx$$

$$2\Omega = \int_0^{\frac{\pi}{2}} \frac{4\sin x + 2\pi + 4\cos x + 2\pi}{\sin x + \cos x + \pi} dx = 4 \int_0^{\frac{\pi}{2}} dx = 2\pi \rightarrow \Omega = \pi$$

1107. Find without any software:

$$\Omega = \int (4\cot^3 x - 5\cot^2 x + 7\cot x) e^x dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Yen Tung Chung-Taichung-Vietnam

$$\begin{aligned} \Omega &= \int (4\cot^3 x - 5\cot^2 x + 7\cot x) e^x dx = \\ &= \int (4\cot x(\csc^2 x - 1) - 5(\csc^2 x - 1) + 7\cot x) e^x dx = \\ &= 4 \int e^x \cdot \cot x \cdot \csc^2 x dx + 3 \int e^x \cdot \cot x dx - 5 \int e^x \cdot \csc^2 x dx + 5 \int e^x dx = \\ &= 4 \left(-\frac{1}{2} e^x \cdot \csc^2 x + \frac{1}{2} \int e^x \cdot \csc^2 x dx \right) + 3 \int e^x \cdot \cot x dx - 5 \int e^x \cdot \csc^2 x dx + 5 \int e^x dx = \\ &= -2e^x \cdot \csc^2 x + 3 \int e^x \cdot \cot x dx - 3 \int e^x \cdot \csc^2 x dx + 5e^x = \\ &= -2e^x(1 + \cot^2 x) + 3 \int (e^x \cot x - e^x \cdot \csc^2 x) dx + 5e^x = \\ &= -2e^x \cdot \cot^2 x + 3e^x \cdot \cot x + 3e^x + C, \text{ where} \end{aligned}$$

$$\begin{cases} u = e^x \cdot \csc x \\ du = e^x \cdot \csc x(1 - \cot x) dx \end{cases} ; \begin{cases} dv = \cot x \cdot \csc x dx \\ v = -\csc x \end{cases}$$

$$\begin{aligned} \int e^x \cdot \cot x \cdot \csc^2 x dx &= -e^x \cdot \csc^2 x + \int e^x \csc^2 x (1 - \cot x) dx = \\ &= -e^x \cdot \csc^2 x + \int e^x \cdot \csc^2 x dx - \int e^x \cdot \cot x \cdot \csc^2 x dx \\ \int e^x \cdot \cot x \cdot \csc^2 x dx &= -\frac{1}{2} e^x \csc^2 x + \frac{1}{2} \int e^x \cdot \csc^2 x dx \end{aligned}$$



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Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\Omega = \int (4\cot^3 x - 5\cot^2 x + 7\cot x)e^x dx$$

Let be the function: $f(x) = (4\cot^3 x - 5\cot^2 x + 7\cot x)e^x$ and

$$F(x) = (A\cot^2 x + B\cot x + C)e^x$$

$$F'(x) = [-2A\cot^3 x + (A - B)\cot^2 x + (-2A + B)\cot x + (-B + C)]e^x = f(x) \Rightarrow \\ -2A = 4 \Rightarrow A = -2; A - B = -5 \Rightarrow B = 3; -B + C = 0 \Rightarrow C = 3.$$

$$\text{So, } F(x) = (-2\cot^3 x + 3\cot x + 3)e^x$$

$$\Omega = \int (4\cot^3 x - 5\cot^2 x + 7\cot x)e^x dx = (-2\cot^3 x + 3\cot x + 3)e^x + C$$

Solution 3 by Adrian Popa-Romania

$$(\cot x)' = -\frac{1}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -(1 + \cot^2 x)$$

$$\Omega = \int (4\cot^3 x - 5\cot^2 x + 7\cot x)e^x dx =$$

$$= \int (4\cot^3 x + 4\cot x - 3\cot^2 x - 3 - 2\cot^2 x + 3\cot x + 3)e^x dx =$$

$$= \int \left[\underbrace{(4\cot x(\cot^2 x + 1) - 3(\cot^2 x + 1))}_{f'(x)} e^x + \underbrace{(-2\cot^2 x + 3\cot x + 3)}_{f(x)} e^x \right] dx =$$

$$= \int (f'(x) + f(x))e^x dx = f(x)e^x + C = (-2\cot^2 x + 3\cot x + 3)e^x + C$$

Solution 4 by Gilmer Lopez-Cajamarca-Peru

$$\Omega = \int (4\cot^3 x - 5\cot^2 x + 7\cot x)e^x dx =$$

$$= \int (4\cot x(\csc^2 x - 1) + 5(\csc^2 x - 1) + 7\cot x)e^x dx =$$

$$= 4 \int \cot x \cdot \csc^2 x \cdot e^x dx - 4 \int \cot x \cdot e^x dx - 5 \int \csc^2 x \cdot e^x dx + 5 \int e^x dx$$

$$+ 7 \int \cot x \cdot e^x dx =$$



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$$\begin{aligned}
 &= 4 \underbrace{\int \cot x \cdot \csc^2 x \cdot e^x dx + 3 \int \cot x \cdot e^x dx}_{\begin{array}{l} u=e^x; \\ du=e^x dx \\ dv=\cot x \cdot \csc^2 x; \\ v=-\frac{1}{2} \cot^2 x \end{array}} - 5 \int \csc^2 x \cdot e^x dx + 5 \int e^x dx = \\
 &= 4 \left(-\frac{1}{2} e^x \cdot \cot^2 x + \frac{1}{2} \int \cot^2 x \cdot e^x dx \right) + 3 \left(\cot x \cdot e^x + \int \csc^2 x \cdot e^x dx \right) \\
 &\quad - 5 \int \csc^2 x \cdot e^x dx + 5 \int e^x dx = \\
 &= -2e^x \cdot \cot^2 x + 2 \int \cot^2 x \cdot e^x dx + 3 \cot x \cdot e^x + 3 \int \csc^2 x \cdot e^x dx \\
 &\quad - 5 \int \csc^2 x \cdot e^x dx + 5 \int e^x dx = \\
 &= -2e^x \cdot \cot^2 x + 2 \int \cot^2 x \cdot e^x dx + 3 \cot x \cdot e^x - 2 \int \csc^2 x \cdot e^x dx + 5 \int e^x dx = \\
 &= -2e^x \cdot \cot^2 x + 2 \int \csc^2 x \cdot e^x dx - 2 \int e^x dx + 3 \cot x \cdot e^x - 2 \int \csc^2 x \cdot e^x dx \\
 &\quad + 5 \int e^x dx = -2e^x \cdot \cot^2 x + 3 \cot x \cdot e^x + 3 \int e^x dx = \\
 &= -2e^x \cdot \cot^2 x + 3 \cot x \cdot e^x + 3e^x + C
 \end{aligned}$$

1108. For $n \geq 1$, prove that:

$$\int_0^\infty \frac{\sqrt[n]{x}}{1+x+x^2+x^3} dx = \frac{\pi}{2} \cdot \frac{1}{\left(1 + \sin \frac{\pi}{2n} + \cos \frac{\pi}{2n}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned}
 \Omega(n) &= \int_0^\infty \frac{\sqrt[n]{x}}{1+x+x^2+x^3} dx = \int_0^\infty \frac{x^{\frac{1}{n}} - x^{\frac{n+1}{n}}}{1-x^4} dx \stackrel{t=x^4}{=} \int_0^\infty \frac{\frac{1}{4}t^{\frac{3}{4}} - t^{\frac{n+1}{4}}}{1-t} dt = \\
 &= \frac{1}{4} \left(\int_0^1 \frac{t^{\frac{1}{4}-\frac{3}{4}} - t^{\frac{1}{4}-\frac{n+1}{2}}}{1-t} dt + \underbrace{\int_1^\infty \frac{t^{\frac{1}{4}-\frac{3}{4}} - t^{\frac{1}{4}-\frac{n+1}{2}}}{1-t} dt}_{z=\frac{1}{t}} \right) =
 \end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{4} \left(\int_0^1 \frac{\frac{1}{t^{4n}} - \frac{3}{4} - t^{\frac{1}{4n}} - \frac{1}{2}}{1-t} dt - \int_0^1 \frac{\frac{1}{z^{-4n}} - \frac{3}{4} - z^{-\frac{1}{4n}} - \frac{1}{2}}{1-z} dz \right) = \\
 &= \frac{1}{4} \left(\Psi\left(\frac{1}{4n} + \frac{1}{2}\right) - \Psi\left(\frac{1}{4n} + \frac{1}{4}\right) - \Psi\left(\frac{1}{2} - \frac{1}{4n}\right) + \Psi\left(\frac{3}{4} - \frac{1}{4n}\right) \right) = \\
 &= \frac{1}{4} \left(\left(\Psi\left(\frac{1}{4n} + \frac{1}{2}\right) - \Psi\left(\frac{1}{2} - \frac{1}{4n}\right) \right) + \left(\Psi\left(\frac{3}{4} - \frac{1}{4n}\right) - \Psi\left(\frac{1}{4} + \frac{1}{4n}\right) \right) \right) \\
 &= \frac{\pi}{4} \left(\cot\left(\frac{\pi}{2} - \frac{\pi}{4n}\right) + \cot\left(\frac{\pi}{4} + \frac{\pi}{4n}\right) \right) = \frac{\pi}{4} \left(\tan\frac{\pi}{4n} + \tan\left(\frac{\pi}{4} - \frac{\pi}{4n}\right) \right) = \\
 &= \frac{\pi}{4} \left(\tan\frac{\pi}{4n} + \frac{1 - \tan\frac{\pi}{4n}}{1 + \tan\frac{\pi}{4n}} \right) = \frac{\pi}{4} \left(\frac{\sin\frac{\pi}{4n}}{\cos\frac{\pi}{4n}} + \frac{\cos\frac{\pi}{4n} - \sin\frac{\pi}{4n}}{\cos\frac{\pi}{4n} + \sin\frac{\pi}{4n}} \right) = \\
 &= \frac{\pi}{2} \cdot \frac{\cos\frac{\pi}{4n} \sin\frac{\pi}{4n} + \sin^2\frac{\pi}{4n} + \cos^2\frac{\pi}{4n} - \cos\frac{\pi}{4n} \sin\frac{\pi}{4n}}{2\cos^2\frac{\pi}{4n} + 2\cos\frac{\pi}{4n} \sin\frac{\pi}{4n}} = \\
 &= \frac{\pi}{2} \cdot \frac{1}{1 + \cos\frac{\pi}{2n} + \sin\frac{\pi}{2n}} \\
 &\int_0^\infty \frac{\sqrt[n]{x}}{1+x+x^2+x^3} dx = \frac{\pi}{2} \cdot \frac{1}{\left(1+\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}\right)}
 \end{aligned}$$

Solution 2 by Tobi Joshua-Nigeria

$$\begin{aligned}
 I &= \int_0^\infty \frac{\sqrt[n]{x}}{1+x+x^2+x^3} dx = \int_0^\infty \frac{\sqrt[n]{x}}{(1+x)(1+x^2)} dx \\
 I &= \frac{1}{2} \int_0^\infty \frac{\sqrt[n]{x}}{1+x} dx - \underbrace{\frac{1}{2} \int_0^\infty \frac{\sqrt[n]{x}(x-1)}{1+x^2} dx}_{y=x^2} = \frac{1}{2} \int_0^\infty \frac{x^{\frac{1}{n}}}{1+x} dx - \frac{1}{4} \int_0^\infty \frac{y^{\frac{1}{2n}} - y^{\frac{1}{2n}-\frac{1}{2}}}{1+y} dy = \\
 &= \frac{1}{2} \int_0^\infty \frac{x^{\left(\frac{1}{n}+1\right)-1}}{1+x} dx - \frac{1}{4} \int_0^\infty \frac{y^{\left(\frac{1}{2n}+\frac{1}{2}\right)-1}}{1+y} dy \\
 &\left(\therefore \int_0^\infty \frac{x^{m-1}}{1+x} dx = \frac{\pi}{\sin(m\pi)} ; m < 0 \right)
 \end{aligned}$$



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$$\begin{aligned}
 I &= \frac{\pi}{2\sin\left(\frac{\pi}{n} + \pi\right)} - \frac{1}{4} \left(\frac{\pi}{\sin\left(\frac{\pi}{2n} + \pi\right)} - \frac{\pi}{\sin\left(\frac{\pi}{2n} + \frac{\pi}{2}\right)} \right) = \\
 &= -\frac{\pi}{2\sin\frac{\pi}{n}} - \frac{1}{4} \left(-\frac{\pi}{\sin\frac{\pi}{2n}} - \frac{\pi}{\cos\frac{\pi}{2n}} \right) = -\frac{\pi}{4\cos\frac{\pi}{2n}\sin\frac{\pi}{2n}} + \frac{1}{4} \cdot \frac{\pi}{\sin\frac{\pi}{2n}} + \frac{1}{4} \cdot \frac{\pi}{\cos\frac{\pi}{2n}} = \\
 &= \frac{\pi}{4} \cdot \frac{\cos\frac{\pi}{2n} + \sin\frac{\pi}{2n} - 1}{\cos\frac{\pi}{2n}\sin\frac{\pi}{2n}} = \frac{\pi}{4} \cdot \frac{\left(\cos\frac{\pi}{2n} + \sin\frac{\pi}{2n}\right)^2 - 1}{\cos\frac{\pi}{2n}\sin\frac{\pi}{2n}\left(\cos\frac{\pi}{2n} + \sin\frac{\pi}{2n} + 1\right)} = \\
 &= \frac{\pi}{4} \cdot \frac{2\cos\frac{\pi}{2n}\sin\frac{\pi}{2n}}{\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}\left(\cos\frac{\pi}{2n} + \sin\frac{\pi}{2n} + 1\right)} \\
 &\int_0^\infty \frac{\sqrt[n]{x}}{1+x+x^2+x^3} dx = \frac{\pi}{2} \cdot \frac{1}{\left(1+\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}\right)}
 \end{aligned}$$

1109. Find without any software:

$$\Omega = \int \frac{3x^2 + x}{1 + 6x(1 + e^{3x}) + 2e^{3x} + e^{6x} + 9x^2} dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned}
 \Omega &= \int \frac{3x^2 + x}{1 + 6x(1 + e^{3x}) + 2e^{3x} + e^{6x} + 9x^2} dx = \int \frac{3x^2 + x}{(3x + e^{3x})^2 + 2(3x + e^{3x}) + 1} dx \\
 &= \int \frac{3x + 1}{(3x + e^{3x} + 1)^2} \cdot x dx = \int \frac{(3x + 1)e^{-3x}}{((3x + 1)e^{-3x} + 1)^2} \cdot xe^{-3x} dx \\
 (t &= (3x + 1)e^{-3x} \Rightarrow dt = (3e^{-3x} - 3(3x + 1)e^{-3x})dx = -9xe^{-3x}dx) \\
 \Omega &= -\frac{1}{9} \int \frac{tdt}{(1+t)^2} = -\frac{1}{9} \left(\log(1+t) + \frac{1}{1+t} \right) + C; C \in \mathbb{R} \\
 \Omega &= -\frac{1}{9} \left(\log(e^{3x} + (3x + 1)) + \frac{e^{3x}}{e^{3x} + (3x + 1)} - 3x \right) + C \\
 \Omega &= \int \frac{3x^2 + x}{1 + 6x(1 + e^{3x}) + 2e^{3x} + e^{6x} + 9x^2} dx =
 \end{aligned}$$



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$$= \frac{x}{3} - \frac{1}{9} \left(\log(e^{3x} + (3x + 1)) + \frac{e^{3x}}{e^{3x} + (3x + 1)} \right) + C$$

Solution 2 by Yen Tung Chung-Taichung-Taiwan

$$\begin{aligned} \Omega &= \int \frac{3x^2 + x}{1 + 6x(1 + e^{3x}) + 2e^{3x} + e^{6x} + 9x^2} dx = \\ &= \int \frac{3x^2 + x}{(1 + e^{3x})^2 + 6x(1 + e^{3x}) + 9x^2} dx = \int \frac{3x^2 + x}{(1 + 3x + e^{3x})^2} dx = \\ &= \int \frac{(3x + 1)e^{-3x}}{((1 + 3x)e^{-3x} + 1)^2} \cdot xe^{-3x} dx \stackrel{y=(1+3x)e^{-3x}+1}{\stackrel{dy=-9xe^{-3x}dx}{=}} \int \frac{y-1}{y} \cdot \left(-\frac{1}{9}dy\right) = \\ &= -\frac{1}{9} \int \left(\frac{1}{y} - \frac{1}{y^2}\right) dy = -\frac{1}{9} \log|y| - \frac{1}{y} + C = \\ &= -\frac{1}{9} \log|(1 + 3x)e^{-3x} + 1| - \frac{1}{(1 + 3x)e^{-3x} + 1} + C = \\ &= -\frac{1}{9} \log \left| \frac{1 + 3x + e^{3x}}{e^{3x}} \right| - \frac{e^{3x}}{1 + 3x + e^{3x}} + C = \\ &= \frac{x}{3} - \frac{1}{9} \cdot \frac{e^{3x}}{1 + 3x + e^{3x}} - \frac{1}{9} \log|1 + 3x + e^{3x}| + C \end{aligned}$$

Solution 3 by Remus Florin Stanca-Romania

$$\begin{aligned} \Omega &= \int \frac{3x^2 + x}{1 + 6x(1 + e^{3x}) + 2e^{3x} + e^{6x} + 9x^2} dx = \\ &= \int \frac{3x^2 + x}{(3x + 1)^2 + (e^{3x} + 3x)^2 - 9x^2 + 2e^{3x}} dx = \\ &= \int \frac{3x^2 + x}{(3x + 1)^2 + (e^{3x} + 3x)^2 + 2e^{3x} + 6x - 9x^2 - 6x - 1 + 1} dx = \\ &= \int \frac{3x^2 + x}{(e^{3x} + 3x)^2 + 2(e^{3x} + 3x) + 1} dx = \int \frac{3x^2 + x}{(e^{3x} + 3x + 1)^2} dx = \\ &= \int \frac{x(e^{3x} + 3x + 1) - xe^{3x}}{(e^{3x} + 3x + 1)^2} dx = \int \frac{x}{e^{3x} + 3x + 1} dx - \int \frac{xe^{3x}}{(e^{3x} + 3x + 1)^2} dx ; (1) \\ &\int \frac{xe^{3x}}{(e^{3x} + 3x + 1)^2} dx = \frac{1}{9} \int \frac{(e^{3x})'(e^{3x} + 3x + 1) - e^{3x}(3e^{3x} + 3)}{(e^{3x} + 3x + 1)^2} dx = \end{aligned}$$



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$$= \frac{1}{9} \cdot \frac{e^{3x}}{e^{3x} + 3x + 1}; (2)$$

$$\begin{aligned} \int \frac{x}{e^{3x} + 3x + 1} dx &= \int \frac{\frac{1}{3}(e^{3x} + 3x + 1) - \frac{1}{9}(3e^{3x} + 3)}{e^{3x} + 3x + 1} dx \\ &= \frac{x}{3} - \frac{1}{9} \log(e^{3x} + 3x + 1); (3) \end{aligned}$$

From (1),(2),(3) we get:

$$\Omega = \frac{x}{3} - \frac{1}{9} \log(e^{3x} + 3x + 1) - \frac{1}{9} \cdot \frac{e^{3x}}{e^{3x} + 3x + 1} + C$$

1110. Find:

$$\Omega = \int \frac{\sqrt[n]{\sin x}}{(\sqrt[n]{\sin x} + \sqrt[n]{\cos x})^{2n+1}} dx, n \in \mathbb{N}, n \geq 2$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Abner Chinga Bazo-Lima-Peru

$$\begin{aligned} \Omega &= \int \frac{\sqrt[n]{\sin x}}{(\sqrt[n]{\sin x} + \sqrt[n]{\cos x})^{2n+1}} dx = \int \frac{\sqrt[n]{\sin x} \cdot (\sqrt[n]{\sec x})^{2n+1}}{(\sqrt[n]{\sin x} + \sqrt[n]{\cos x})^{2n+1} \cdot (\sqrt[n]{\sec x})^{2n+1}} dx = \\ &= \int \frac{\sqrt[n]{\tan x} \cdot \sec^2 x}{(\sqrt[n]{\tan x} + 1)^{2n+1}} dx \stackrel{\sec^2 x dx = nu^{n-1} du}{=} n \int \frac{u^n}{(u+1)^{2n+1}} du \stackrel{u+1=t}{=} \\ &= \int \frac{(t-1)^n}{t^{2n+1}} dt = \int \frac{\sum_{k=0}^n \binom{n}{k} t^{n-k} (-1)^k}{t^{2n+1}} dt = \\ &= \int \frac{1}{t^{2n+1}} \left[\binom{n}{0} t^n - \binom{n}{1} t^{n-1} + \binom{n}{2} t^{n-2} - \binom{n}{3} t^{n-3} + \dots + (-1)^k \binom{n}{n} t^{-2n-1} \right] dt = \\ &= \int \left[\binom{n}{0} t^{-n-1} - \binom{n}{1} t^{-n-2} + \binom{n}{2} t^{-n-3} - \binom{n}{3} t^{-n-4} + \dots + (-1)^k \binom{n}{n} t^{-2n-1} \right] dt = \\ &= \left[\binom{n}{0} \frac{t^{-n}}{n} - \binom{n}{1} \frac{t^{-n-1}}{n+1} + \binom{n}{2} \frac{t^{-n-2}}{n+2} - \binom{n}{3} \frac{t^{-n-3}}{n+3} + \dots + (-1)^k \binom{n}{n} \frac{t^{-2n}}{2n} \right] + C \end{aligned}$$



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1111. Find without any software:

$$\Omega = \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) \sqrt{(1 + \sin x)(1 + \cos x)} dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Abner Chinga Bazo-Lima-Peru

$$\begin{aligned} \Omega &= \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) \sqrt{(1 + \sin x)(1 + \cos x)} dx = \\ &= \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) \sqrt{4\sin^2\left(\frac{x}{2} + \frac{\pi}{4}\right) \cos^2\frac{x}{2}} dx = \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) 2\sin\left(\frac{x}{2} + \frac{\pi}{4}\right) \cos\frac{x}{2} dx = \\ &= \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) \left(\sin\left(x + \frac{\pi}{4}\right) + \sin\frac{\pi}{4}\right) dx = \\ &= \int_0^{\frac{\pi}{4}} \left[\sin\left(x - \frac{\pi}{4}\right) \sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) \sin\frac{\pi}{4}\right] dx = \\ &= -\frac{1}{2} \int_0^{\frac{\pi}{4}} \left[\cos 2x - \cos\frac{\pi}{2} + \cos x - \cos\left(\frac{\pi}{2} - x\right)\right] dx = \\ &= -\frac{1}{2} \int_0^{\frac{\pi}{4}} (\cos 2x + \cos x - \sin x) dx = -\frac{1}{2} \left[\frac{1}{2} \sin 2x + \sin x + \cos x \right]_0^{\frac{\pi}{4}} = -\frac{1}{4} (2\sqrt{2} - 1) \end{aligned}$$

Solution 2 by Abner Chinga Bazo-Lima-Peru

$$\Omega = \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) \sqrt{(1 + \sin x)(1 + \cos x)} dx =$$



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$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) \sqrt{4\sin^2\left(\frac{x}{2} + \frac{\pi}{4}\right) \cos^2\frac{x}{2}} dx = \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) 2\sin\left(\frac{x}{2} + \frac{\pi}{4}\right) \cos\frac{x}{2} dx = \\
 &= \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) \left(\sin\left(x + \frac{\pi}{4}\right) + \sin\frac{\pi}{4}\right) dx = \\
 &= \int_0^{\frac{\pi}{4}} \left[\sin\left(x - \frac{\pi}{4}\right) \sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) \sin\frac{\pi}{4}\right] dx = \\
 &= \int_0^{\frac{\pi}{4}} \left[\sin\left(x - \frac{\pi}{4}\right) \cos\left(x - \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) \sin\frac{\pi}{4}\right] dx = \\
 &= \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) d\left(\sin\left(x - \frac{\pi}{4}\right)\right) + \sin\frac{\pi}{4} \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) d\left(x - \frac{\pi}{4}\right) = \\
 &= \frac{1}{2} \sin^2\left(x - \frac{\pi}{4}\right) \Big|_0^{\frac{\pi}{4}} - \frac{\sqrt{2}}{2} \cos\left(x - \frac{\pi}{4}\right) \Big|_0^{\frac{\pi}{4}} = -\frac{1}{4}(2\sqrt{2} - 1)
 \end{aligned}$$

Solution 3 by Igor Soposki-Skopje-Macedonia

$$\begin{aligned}
 \Omega &= \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4}\right) \sqrt{(1 + \sin x)(1 + \cos x)} dx \stackrel{x - \frac{\pi}{4} = t}{=} \\
 &= \int_{-\frac{\pi}{4}}^0 \sin t \sqrt{(1 + \sin(t + \frac{\pi}{4}))(1 + \cos(t + \frac{\pi}{4}))} dt = \\
 &= \int_{-\frac{\pi}{4}}^0 \sin t \sqrt{\left(1 + \frac{\sqrt{2}}{2}(\sin t + \cos t)\right) \left(1 + \frac{\sqrt{2}}{2}(\cos t - \sin t)\right)} dt = \\
 &= \int_{-\frac{\pi}{4}}^0 \sin t \sqrt{\frac{1}{2} + \sqrt{2}\cos t + \cos^2 t} dt = \int_{-\frac{\pi}{4}}^0 \sin t \sqrt{\left(\cos t + \frac{1}{\sqrt{2}}\right)^2} dt =
 \end{aligned}$$



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$$\begin{aligned}
 &= \int_{-\frac{\pi}{4}}^0 \sin t \left(\cos t + \frac{1}{\sqrt{2}} \right) dt \stackrel{u=\cos t}{=} - \int_{\frac{\sqrt{2}}{2}}^1 \left(u + \frac{1}{\sqrt{2}} \right) du = \\
 &= - \left[\frac{t^2}{2} + \frac{t}{\sqrt{2}} \right]_{\frac{\sqrt{2}}{2}}^1 = \frac{1}{4} - \frac{1}{\sqrt{2}}
 \end{aligned}$$

1112. $f: (0, \infty) \rightarrow \mathbb{R}$, f –continuous, $f(x) - \log_3 x = 4 - f(5^{\log_3 x})$, $\forall x > 0$.

Find:

$$\Omega = \int_2^3 (f(x) - 2) \cdot \log_x 15 dx$$

Proposed by Daniel Sitaru-Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$f(x) - \log_3 x = 4 - f(5^{\log_3 x}); \quad (1)$$

Put $f(x) = g(x) + 2 + \log_{15} x$, we have

$$\begin{aligned}
 f(5^{\log_3 x}) &= g(5^{\log_3 x}) + 2 + \log_{15}(5^{\log_3 x}) = g(5^{\log_3 x}) + 2 + \log_3 x \cdot \log_{15} 5 = \\
 &= g(5^{\log_3 x}) + 2 + \log_3 x (1 - \log_{15} 3)
 \end{aligned}$$

So, the equation (1) is equivalent to:

$$g(x) + 2 + \log_{15} x - \log_3 x = 4 - g(5^{\log_3 x}) - 2 - \log_3 x (1 - \log_{15} 3)$$

$$\text{Or: } g(x) + 2 + \log_3 x (\log_{15} 3 - 1) = 4 - g(5^{\log_3 x}) - 2 + \log_3 x (\log_{15} 3 - 1) \Rightarrow$$

$$g(x) = -g(5^{\log_3 x}); \quad (2)$$

Sobstitute $x \rightarrow 3^{\log_5 x}$, we have the functional equation (2) equivalent to:

$$g(3^{\log_5 x}) = -g(x) \Rightarrow g(x) = -g(3^{\log_5 x}); \quad (3)$$

Substitute $x \rightarrow 3^{\log_5 x}$, we have the functional equation (3) equivalent to:

$$g(3^{\log_5 x}) = -g(3^{\log_5 3 \cdot \log_5 x}); \quad (4)$$

From (3),(4) we have: $g(x) = (-1)^2 g(3^{\log_5 3 \cdot \log_5 x})$

Similarly, we have:

$$g(x) = (-1)^1 g(3^{\log_5 x}) = (-1)^2 g(3^{\log_5 3 \cdot \log_5 x}) = (-1)^3 g(3^{\log_5^2 3 \cdot \log_5 x}) = \dots =$$



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$$= (-1)^{2n} g\left(3^{\log^{2n-1} \frac{1}{5} 3 \cdot \log_5 x}\right); \forall n \geq 1$$

$$\text{So, we have: } g(x) = g\left(3^{\log^{2n-1} \frac{1}{5} 3 \cdot \log_5 x}\right)$$

$$\text{Given } n \rightarrow \infty, \text{ we have } g(x) = g(1)$$

Substitute $x = 1$ in (2), we have $g(1) = -g(1) \Rightarrow g(1) = 0 \Rightarrow f(x) = 2 + \log_{15} x$

So, we have:

$$\begin{aligned} \Omega &= \int_2^3 (f(x) - 2) \cdot \log_x 15 dx = \int_2^3 (2 + \log_{15} x - 2) \cdot \log_x 15 dx = \\ &= \int_2^3 \log_{15} x \cdot \log_x 15 dx = 1 \end{aligned}$$

1113. Find without any software:

$$\Omega = \int_{\frac{\pi^5}{1024}}^{\frac{\pi^5}{243}} \frac{\sin(\sqrt[5]{x}) \cdot \sin(5\sqrt[5]{x}) \cdot \sin(5^5\sqrt[5]{x})}{\sqrt[5]{x^4}} dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Abner Chinga Bazo-Lima-Peru

$$\begin{aligned} \Omega &= \int_{\frac{\pi^5}{1024}}^{\frac{\pi^5}{243}} \frac{\sin(\sqrt[5]{x}) \cdot \sin(5\sqrt[5]{x}) \cdot \sin(5^5\sqrt[5]{x})}{\sqrt[5]{x^4}} dx \stackrel{\sqrt[5]{x}=t}{=} 5 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin t \cdot \sin 3t \cdot \sin 5t dt = \\ &= \frac{5}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\cos 2t - \cos 4t) \sin 5t dt = \frac{5}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin 7t + \sin 3t - \sin 9t - \sin t) dt = \\ &= \frac{5}{4} \left[-\frac{\cos 7t}{7} - \frac{\cos 3t}{3} + \frac{\cos 9t}{9} + \cos t \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{5}{252} (41 - 41\sqrt{2}) = \frac{205}{252} (1 - \sqrt{2}) \end{aligned}$$

Solution 2 by Timson Azeem Folorunsho-Lagos-Nigeria



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$$\begin{aligned}
 \Omega &= \int_{\frac{\pi^5}{1024}}^{\frac{\pi^5}{243}} \frac{\sin(\sqrt[5]{x}) \cdot \sin(5\sqrt[5]{x}) \cdot \sin(5\sqrt[5]{x})}{\sqrt[5]{x^4}} dx \stackrel{\sqrt[5]{x}=y}{=} 5 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin y \cdot \sin 3y \cdot \sin 5y dy = \\
 &= 5 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\cos 2y - \cos 8y}{2} \right) \sin y dy = \frac{5}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\cos 2y \sin y - \cos 8y \sin y) dy = \\
 &= \frac{5}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 3y - \sin y - \sin 9y + \sin 7y}{2} dy = \frac{5}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin 3y - \sin y - \sin 9y + \sin 7y) dy \\
 &= \frac{5}{4} \left[-\frac{\cos 7t}{7} - \frac{\cos 3t}{3} + \frac{\cos 9t}{9} + \cos t \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{5}{4} \left(\frac{41}{63} - \frac{41\sqrt{2}}{63} \right) = \frac{205 - 205\sqrt{2}}{252}
 \end{aligned}$$

1114. Find a closed form:

$$\Omega(a) = \int_0^\infty \frac{x^3}{(x^4 - x^2 + 1)(ax + 1)} dx, a > 0$$

Proposed by Vasile Mircea Popa-Romania

Solution by Kamel Benaicha-Algiers-Algerie

$$\frac{x^3}{(x^4 - x^2 + 1)(ax + 1)} = \frac{A}{ax + 1} + \frac{Bx^3 + Cx^2 + Dx + E}{x^4 - x^2 + 1}; \quad (1)$$

$$(1) \cdot (ax + 1), x = -\frac{1}{a} \Rightarrow A = -\frac{a}{a^4 - a^2 + 1}$$

$$(1) \cdot x, x \rightarrow \infty \Rightarrow \frac{A}{a} + B = 0 \Rightarrow B = \frac{1}{a^4 - a^2 + 1}$$

$$x = 0 \Rightarrow E = -A = \frac{a}{a^4 - a^2 + 1}$$

$$x = 1 \Rightarrow B + C + D + E + \frac{A}{1+a} = \frac{1}{1+a} \Rightarrow$$

$$C + D = \frac{1}{1+a} \left(1 - \frac{(1+a)^2}{a^4 - a^2 + 1} + \frac{a}{a^4 - a^2 + 1} \right) = \frac{a^4 - 2a^2 + 1}{(1+a)(a^4 - a^2 + 1)}$$



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$$x = 1 \Rightarrow -B + C - D + E + \frac{A}{1-a} = -\frac{1}{1+a} \Rightarrow$$

$$C - D = \frac{1}{1-a} \left(-1 + \frac{(1-a)^2}{a^4 - a^2 + 1} + \frac{a}{a^4 - a^2 + 1} \right) = -\frac{a^4 - 2a^2 + 1}{(1-a)(a^4 - a^2 + 1)}$$

$$\begin{cases} C + aD = -\frac{a}{a^4 - a^2 + 1} \\ aC + D = \frac{a^2 - 2a}{a^4 - a^2 + 1} \end{cases} \Rightarrow \begin{cases} C = \frac{a(a^2 - 1)}{a^4 - a^2 + 1} \\ D = -\frac{a^2}{a^4 - a^2 + 1} \end{cases}$$

$$\begin{aligned} \Omega(a) &= \frac{1}{a^4 - a^2 + 1} \int_0^\infty \frac{-ax}{1+ax} dx + \frac{1}{4(a^4 - a^2 + 1)} \int_0^\infty \frac{4x^3 - 2x}{x^4 - x^2 + 1} dx + \\ &+ \underbrace{\frac{a(a^2 - 1)}{a^4 - a^2 + 1} \int_0^\infty \frac{x^2}{x^4 - x^2 + 1} dx}_{t=1/x} + \underbrace{\frac{(1 - 2a^2)}{2(a^4 - a^2 + 1)} \int_0^\infty \frac{x}{x^4 - x^2 + 1} dx}_{z=x^2} + \\ &+ \frac{a}{a^4 - a^2 + 1} \int_0^\infty \frac{1}{x^4 - x^2 + 1} dx = \\ &= \frac{1}{a^4 - a^2 + 1} \left(-\log a + a^3 \int_0^\infty \frac{1+t^2}{1+t^6} dt + \frac{1-2a^2}{4} \int_0^\infty \frac{dz}{z^2 - z + 1} \right) = \\ &= \frac{1}{a^4 - a^2 + 1} \left(-\log a + \frac{a^3}{6} \int_0^\infty \frac{u^{-\frac{5}{6}} + u^{-\frac{1}{2}}}{1+u} du + \frac{1-2a^2}{4} \int_0^\infty \frac{dz}{\left(z - \frac{1}{2}\right)^2 + \frac{3}{4}} \right) = \\ &= \frac{1}{a^4 - a^2 + 1} \left(-\log a + \frac{a^3 \pi}{6} \left(\frac{1}{\sin \frac{5\pi}{6}} + \frac{1}{\sin \frac{\pi}{2}} \right) + \frac{1-2a^2}{2\sqrt{3}} \tan^{-1} \left(\frac{2z}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right) \Big|_0^\infty \right) = \\ &= \frac{1}{a^4 - a^2 + 1} \left(-\log a + \frac{a^3 \pi}{2} + \frac{1-2a^2}{2\sqrt{3}} \left(\frac{\pi}{2} - \frac{\pi}{6} \right) \right) = \\ &= \frac{1}{a^4 - a^2 + 1} \left(-\log a + \left(\frac{a^3}{2} + \frac{1-2a^2}{2\sqrt{3}} \right) \pi \right) \\ \Omega(a) &= \int_0^\infty \frac{x^3}{(x^4 - x^2 + 1)(ax + 1)} dx = \frac{3\sqrt{3}a^3 - 4a^2 + 2}{6\sqrt{3}(a^4 - a^2 + 1)} \cdot \pi - \frac{\log a}{a^4 - a^2 + 1} \end{aligned}$$



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1115. Evaluate the following integral in a closed form:

$$\int_0^1 \left(\int_0^\infty \frac{\tan^{-1}(\sqrt[4]{x})}{xy + 1} \cdot \frac{dx}{x} \right) dy$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Kamel Benaicha-Algiers-Algeria

$$\begin{aligned}
\Omega &= \int_0^1 \left(\int_0^\infty \frac{\tan^{-1}(\sqrt[4]{x})}{xy + 1} \cdot \frac{dx}{x} \right) dy \stackrel{t=\sqrt[4]{x}}{=} 4 \int_0^1 \int_0^\infty \frac{\tan^{-1}t}{t(1+yt^4)} dt dy = \\
&= \int_0^1 \left(\frac{\pi}{2} \log\left(\frac{1}{y}\right) - \int_0^\infty \frac{\log(t^4) - \log(1+yt^4)}{1+t^2} dt \right) dy = \\
&= \frac{\pi}{2} + \int_0^1 \int_0^\infty \frac{\log(1+yt^4)}{1+t^2} dt dy = \frac{\pi}{2} + \int_0^\infty \frac{(1+t^4)\log(1+t^4) - t^4}{(1+t^2)t^4} dt = \\
&= \int_0^\infty \frac{(1+t^4)\log(1+t^4)}{(1+t^2)t^4} dt \stackrel{z=\frac{1}{t^2}}{=} \frac{1}{2} \int_0^\infty \frac{(1+z^2)(\log(1+z^2) - 2\log z)}{(1+z)\sqrt{z}} dz = \\
&= \frac{1}{2} \int_0^\infty \frac{(1+z^2)(\log(1+z^2))}{(1+z)z^{\frac{5}{2}}} dz \\
\text{Put: } I(\alpha) &= \frac{1}{2} \int_0^\infty \frac{(1+z^2)\log(1+\alpha z^2)}{(1+z)z^{\frac{5}{2}}} dz \\
I'(\alpha) &= \frac{1}{2} \int_0^\infty \frac{1+z^2}{(1+z)(1+\alpha z^2)z^{\frac{1}{2}}} dz = \\
&= \frac{1}{2(1+\alpha)} \int_0^\infty \left(\frac{1+z^2}{(1+z)\sqrt{z}} + \frac{(1-z)(\alpha-1+1+\alpha z^2)}{(1+\alpha z^2)\sqrt{z}} \right) dz = \\
&= \frac{1}{2(1+\alpha)} \int_0^\infty \left(\frac{1+z}{\sqrt{z}} + \frac{-2z}{(1+z)\sqrt{z}} + \frac{1-z}{\sqrt{z}} + \frac{(1-z)(\alpha-1)}{(1+\alpha z^2)\sqrt{z}} \right) dz =
\end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{2(1+\alpha)} \int_0^\infty \left(\frac{2}{\sqrt{z}} + \frac{-2z}{(1+z)\sqrt{z}} + \frac{(1-z)(\alpha-1)}{(1+\alpha z^2)\sqrt{z}} \right) dz = \\
 &= \frac{1}{2(1+\alpha)} \int_0^\infty \left(\frac{2}{(1+z)\sqrt{z}} - (1-\alpha) \cdot \frac{1-z}{(1+\alpha z^2)\sqrt{z}} \right) dz \stackrel{u=\sqrt{z}}{=} \\
 &= \frac{1}{2(1+\alpha)} \left(4 \int_0^\infty \frac{du}{1+u^2} - 2(1-\alpha) \int_0^\infty \frac{1-u^2}{1+\alpha u^2} du \right) \stackrel{t=\alpha u^4}{=} \\
 &= \frac{1}{1+\alpha} \left(\pi - \frac{1}{4}(1-\alpha)\alpha^{-\frac{1}{4}} \int_0^\infty \frac{t^{-\frac{3}{4}} - t^{-\frac{1}{4}}}{1+t} dt \right) \\
 \Omega &= \pi \log 2 - \frac{\pi}{2\sqrt{2}} \int_0^1 \frac{(1-\alpha)(\sqrt{\alpha}-1)\alpha^{-\frac{3}{4}}}{1+\alpha} d\alpha = \\
 &= \pi \log 2 - \frac{\pi}{2\sqrt{2}} \int_0^1 \frac{\alpha^{-\frac{3}{4}}(\sqrt{\alpha}-\alpha\sqrt{\alpha}+\alpha-1)(1-\alpha)}{1-\alpha^2} d\alpha \stackrel{t=\alpha^2}{=} \\
 &= \pi \log 2 - \frac{\pi}{4\sqrt{2}} \int_0^1 \frac{t^{-\frac{7}{8}}(t^{\frac{1}{4}}-t^{\frac{3}{4}}+t^{\frac{1}{2}}-1)(1-t^{\frac{1}{2}})}{1-t} dt = \\
 &= \pi \log 2 - \frac{\pi}{4\sqrt{2}} \int_0^1 \frac{t^{-\frac{5}{8}}-t^{-\frac{1}{8}}+t^{-\frac{3}{8}}-t^{-\frac{7}{8}}-t^{-\frac{1}{8}}+t^{\frac{3}{8}}-t^{\frac{1}{8}}+t^{-\frac{3}{8}}}{1-t} dt = \\
 &= \pi \log 2 - \frac{\pi}{4\sqrt{2}} \int_0^1 \frac{t^{-\frac{5}{8}}-2t^{-\frac{1}{8}}+2t^{-\frac{3}{8}}-t^{-\frac{7}{8}}+t^{\frac{3}{8}}-t^{\frac{1}{8}}}{1-t} dt = \\
 &= \pi \log 2 - \frac{\pi}{4\sqrt{2}} \left(\Psi\left(\frac{1}{8}\right) - \Psi\left(\frac{3}{8}\right) + 2 \left(\Psi\left(\frac{7}{8}\right) - \Psi\left(\frac{5}{8}\right) \right) + \Psi\left(\frac{1}{8}+1\right) - \Psi\left(\frac{3}{8}+1\right) \right) = \\
 &= \pi \log 2 - \frac{\pi}{2\sqrt{2}} \left(\Psi\left(\frac{1}{8}\right) - \Psi\left(\frac{3}{8}\right) - \Psi\left(\frac{1}{2}+\frac{1}{8}\right) + \Psi\left(\frac{1}{2}+\frac{3}{8}\right) + \frac{8}{3} \right) = \\
 &= \pi \log 2 - \frac{\pi}{2\sqrt{2}} \left(2\sqrt{2} \log \left(\frac{2-\sqrt{2}}{2+\sqrt{2}} \right) + \frac{8}{3} \right) =
 \end{aligned}$$



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$$= \pi \log 2 - \pi \log \left(\frac{2 - \sqrt{2}}{2 + \sqrt{2}} \right) - \frac{4\pi}{2\sqrt{2}} = 2\pi \log(2 + \sqrt{2}) - \frac{4\pi}{2\sqrt{2}}$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\begin{aligned}
& \int_0^1 \left(\int_0^\infty \frac{\tan^{-1}(\sqrt[4]{x})}{xy+1} \cdot \frac{dx}{x} \right) dy \stackrel{z=\sqrt[4]{x}}{=} 4 \int_0^1 \int_0^\infty \frac{\tan^{-1} z}{z(1+yz^4)} dz dy = \\
& = 4 \int_0^1 \left(\int_0^\infty \frac{\tan^{-1} z}{z(1+yz^4)} dz \right) dy = 4 \int_0^\infty \frac{\tan^{-1} x \cdot \log(1+x^4)}{x^5} dx \stackrel{IBP}{=} \\
& = \underbrace{\left[4 \left(\log x - \frac{(1+x^4)\log(1+x^4)}{4x^4} \right) \tan^{-1} x \right]_0^\infty}_{=0} - 4 \underbrace{\int_0^\infty \frac{\log x}{1+x^2} dx}_{=0} + \\
& + \int_0^\infty \frac{(1+x^4)\log(1+x^4)}{x^4(1+x^2)} dx = \int_0^\infty \left(\frac{1}{x^4} - \frac{1}{x^2} + \frac{2}{1+x^2} \right) \log(1+x^4) dx = \\
& = \pi \left(\frac{\sqrt{2}}{3} - \sqrt{2} \right) + 2 \int_0^\infty \int_0^1 \frac{x^4}{(1+x^2)(1+mx^4)} dm dx = \\
& = -\frac{\pi\sqrt{2}}{3} + 2 \int_0^\infty \int_0^1 \frac{1}{m+1} \left(\frac{1}{1+x^2} + \frac{1-x^2}{m+x^4} \right) dx dm = \\
& = -\frac{\pi\sqrt{2}}{3} + 2 \int_0^1 \frac{1}{m+1} \left(\frac{\pi}{2\sqrt{2}\sqrt[4]{m^3}} - \frac{\pi}{2\sqrt{2}\sqrt[4]{m}} + \frac{\pi}{2} \right) dm = \\
& = \pi \left(-\frac{2\sqrt{2}}{3} + \log 2 + 2\sqrt{2} \int_0^1 \frac{1-m^2}{1+m^4} dm \right) = \pi \left(-\frac{2\sqrt{2}}{3} + \log 2 + 2 \operatorname{argcoth}(\sqrt{2}) \right) \\
& \int_0^1 \left(\int_0^\infty \frac{\tan^{-1}(\sqrt[4]{x})}{xy+1} \cdot \frac{dx}{x} \right) dy = \pi \left(\log(6+4\sqrt{2}) - \frac{2\sqrt{2}}{3} \right) \\
& \text{Note: } \int_0^\infty \frac{\log(1+x)}{x^{1-p}} dx = \frac{\pi}{psin(p\pi)}, -1 < Re(p) < 0
\end{aligned}$$



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1116. Find:

$$\Omega(a) = \int_1^2 \frac{\log(ax)}{x^2 + 3x + 2} dx, a \geq 1$$

Proposed by Radu Diaconu-Romania

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned}
x^2 + 3x + 2 &= \left(x + \frac{3}{2}\right)^2 - \frac{1}{4} = (x+1)(x+2) \\
\Omega(a) &= \int_1^2 \frac{\log(ax)}{x^2 + 3x + 2} dx = \log a \int_1^2 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx + \int_1^2 \frac{\log x}{x+1} dx - \int_1^2 \frac{\log x}{x+2} dx = \\
&= \log a \cdot \log\left(\frac{9}{8}\right) + \log 2 \cdot \log\left(\frac{3}{4}\right) - \underbrace{\int_1^2 \frac{\log(x+1)}{x} dx}_{t=\frac{1}{x}} + \underbrace{\int_1^2 \frac{\log(2+x)}{x} dx}_{t=\frac{1}{x}} = \\
&= \log a \cdot \log\left(\frac{9}{8}\right) + \log 2 \cdot \log\left(\frac{3}{4}\right) + \frac{1}{2} \log^2 2 - \int_{\frac{1}{2}}^1 \frac{\log(1+t)}{t} dt + \int_1^2 \frac{\log\left(1+\frac{x}{2}\right)}{x} dx = \\
&= \log\left(\frac{3}{4}\right) \cdot \log(2a) + \log a \cdot \log\left(\frac{3}{2}\right) + \frac{\log^2 2}{2} + \\
&+ \left(Li_2(-1) - Li_2\left(-\frac{1}{2}\right) - Li_2(-1) + Li_2\left(-\frac{1}{2}\right)\right) = \\
&= \log\left(\frac{3}{4}\right) \cdot \log(2a) + \log a \cdot \log\left(\frac{3}{2}\right) + \frac{\log^2 2}{2}
\end{aligned}$$

Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$\Omega(a) = \int_1^2 \frac{\log(ax)}{x^2 + 3x + 2} dx = \underbrace{\int_1^2 \frac{\log a}{x^2 + 3x + 2} dx}_{P_1} + \underbrace{\int_1^2 \frac{\log x}{x^2 + 3x + 2} dx}_{P_2} =$$

We have:

$$P_1 = \int_1^2 \frac{\log a}{x^2 + 3x + 2} dx = \log a \int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \log a \cdot \log\left(\frac{x+1}{x+2}\right) \Big|_1^2 =$$



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$$= \log a \cdot \left(\log \frac{3}{4} - \log \frac{2}{3} \right) = \log a \cdot \log \frac{9}{8}$$

And on the other hand, we have:

$$\begin{aligned} P_2 &= \int_1^2 \frac{\log x}{x^2 + 3x + 2} dx = \int_1^2 \left(\frac{1}{x+1} - \frac{1}{x+2} \right) \log x dx = \int_1^2 \log x \cdot \left(\log \left(\frac{x+1}{x+2} \right) \right)' dx = \\ &= \log x \cdot \left(\log \left(\frac{x+1}{x+2} \right) \right) \Big|_1^2 - \int_1^2 \frac{\log(x+1) - \log(x+2)}{x} dx = \end{aligned}$$

$$= \log 2 \cdot \log \frac{3}{4} - \underbrace{\int_1^2 \frac{\log(x+1)}{x} dx}_{P_3} + \underbrace{\int_1^2 \frac{\log(x+2)}{x} dx}_{P_4}$$

$$\begin{aligned} P_3 &= \int_1^2 \frac{\log(x+1)}{x} dx \stackrel{x=-t}{=} \int_{-1}^{-2} \frac{\log(1-t)}{-t} \cdot (-dt) = \int_{-2}^{-1} -\frac{\log(1-t)}{t} dt = Li_2(t)|_{-2}^{-1} = \\ &= Li_2(-1) - Li_2(-2) \end{aligned}$$

$$\begin{aligned} P_4 &= \int_1^2 \frac{\log(x+2)}{x} dx = \int_1^2 \frac{\log(x+2) - \log 2 + \log 2}{x} dx = \int_1^2 \frac{\log \left(\frac{x}{2} + 1 \right) + \log 2}{x} dx \\ &= \frac{1}{2} \int_1^2 \frac{\log \left(\frac{x}{2} + 1 \right)}{\frac{x}{2}} dx + \int_1^2 \frac{\log 2}{x} dx = \int_{\frac{1}{2}}^1 \frac{\log(t+1)}{t} dt + \int_1^2 \frac{\log 2}{x} dx = \\ &= -Li_2(-t)|_{\frac{1}{2}}^1 + \log 2 \cdot \log x|_1^2 = Li_2(-1) + Li_2\left(-\frac{1}{2}\right) + \log^2 2 \end{aligned}$$

So, we have:

$$\begin{aligned} P_2 &= \log 2 \cdot \log \frac{3}{4} - Li_2(-1) + Li_2(-2) - Li_2(-1) + Li_2\left(-\frac{1}{2}\right) + \log^2 2 = \\ &= Li_2(-2) + Li_2\left(-\frac{1}{2}\right) - 2Li_2(-1) + \log 2 \cdot \log \frac{3}{2} = \\ &= Li_2(-2) + Li_2\left(-\frac{1}{2}\right) + \frac{\pi^2}{6} + \log 2 \cdot \log \frac{3}{2} \end{aligned}$$

$$\Omega(a) = \log a \cdot \log \frac{9}{8} + Li_2(-2) + Li_2\left(-\frac{1}{2}\right) + \frac{\pi^2}{6} + \log 2 \cdot \log \frac{3}{2} =$$



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$$= \log a \cdot \log \frac{9}{8} - \frac{\log^2 2}{2} + \log 2 \cdot \log \frac{3}{2}$$

Solution 3 by Zaharia Burgheloa-Romania

$$\begin{aligned}\Omega(a) &= \int_1^2 \frac{\log(ax)}{x^2 + 3x + 2} dx \stackrel{x \rightarrow \frac{2}{x}}{=} \int_1^2 \frac{\log\left(\frac{2a}{x}\right)}{x^2 + 3x + 2} dx \\ 2\Omega(a) &= \int_1^2 \frac{\log(ax) + \log\left(\frac{2a}{x}\right)}{x^2 + 3x + 2} dx = \log(2a^2) \int_1^2 \frac{1}{(x+1)(x+2)} dx \\ \Omega(a) &= \frac{\log(2a^2)}{2} \cdot \log\left(\frac{x+1}{x+2}\right) \Big|_1^2 = \frac{\log(2a^2)}{2} \cdot \log\left(\frac{9}{8}\right), a \geq 1\end{aligned}$$

1117. Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x \cdot \sec^4 x}{e^{2\pi \tan x} - 1} dx = 2 \int_0^{\frac{\pi}{2}} \frac{\sin 2x \cdot \sec^4 x}{e^{2\pi \tan x} + 1} dx = \frac{1}{12}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Benaicha-Algiers-Algerie

$$\text{Put } t = \tan x \Rightarrow dx = \frac{dt}{1+t^2}, \sec^4 x = (1+t^2)^2, \sin 2x = 2\sqrt{1-\cos^2 x} \cdot \cos x = \frac{2t}{1+t^2}$$

$$\Omega_1 = \int_0^{\frac{\pi}{2}} \frac{\sin 2x \cdot \sec^4 x}{e^{2\pi \tan x} - 1} dx = 2 \int_0^{\infty} \frac{tdt}{e^{2\pi t} - 1} \stackrel{2\pi t = z}{=} \frac{1}{2\pi^2} \int_0^{\infty} \frac{z}{e^z - 1} dz = \frac{\zeta(2)\Gamma(2)}{2\pi^2} = \frac{1}{12}$$

With the same way, we get:

$$\begin{aligned}\Omega_2 &= 2 \int_0^{\frac{\pi}{2}} \frac{\sin 2x \cdot \sec^4 x}{e^{2\pi \tan x} + 1} dx = \int_0^{\infty} \frac{tdt}{e^{2\pi t} + 1} \stackrel{2\pi t = z}{=} \frac{1}{\pi^2} \int_0^{\infty} \frac{z}{e^z + 1} dz = \frac{\eta(2)\Gamma(2)}{\pi^2} \\ &= \frac{1 - 2^{1-2}}{\pi^2} \zeta(2) = \frac{1}{12}\end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x \cdot \sec^4 x}{e^{2\pi \tan x} - 1} dx = 2 \int_0^{\frac{\pi}{2}} \frac{\sin 2x \cdot \sec^4 x}{e^{2\pi \tan x} + 1} dx = \frac{1}{12}$$



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Note:

$$\int_0^\infty \frac{z}{e^z - 1} dz = \sum_{n=0}^\infty \int_0^\infty z e^{-(n+1)z} dz = \sum_{n=0}^\infty \frac{1}{(n+1)^2} \int_0^\infty ue^{-u} du = \zeta(2)\Gamma(2) = \frac{\pi^2}{6}$$

$$\int_0^\infty \frac{z}{e^z + 1} dz = \sum_{n=0}^\infty (-1)^n \int_0^\infty z e^{-(n+1)z} dz = \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2} \int_0^\infty ue^{-u} du = \eta(2)\Gamma(2) = \frac{\pi^2}{12}$$

1118.

If $R(k) = \int_0^\infty \frac{x^{-ik}}{x^2 - x + 1} dx$ then prove that:

$$\int_0^\infty R(k) dk = \pi$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} R(k) &= \int_0^\infty \frac{x^{-ik}}{x^2 - x + 1} dx = \int_0^\infty \frac{x^{-ik} + x^{-ik+1}}{1 + x^3} dx = \frac{1}{3} \int_0^\infty \frac{t^{-\frac{ik+2}{3}} + t^{-\frac{ik+1}{3}}}{1 + t} dt = \\ &= \frac{\pi}{3} \left(\frac{1}{\sin\left(\frac{(ik+2)\pi}{3}\right)} + \frac{1}{\sin\left(\frac{(ik+1)\pi}{3}\right)} \right) \end{aligned}$$

$$\text{We have: } \int \frac{dt}{\sin(at+b)} = \frac{1}{a} \log \left(\left| \tan\left(\frac{at+b}{2}\right) \right| \right) + C$$

$$\int_0^\infty R(k) dk = \frac{1}{i} \left[\log \left(\left| \tan\left(\frac{(ik+2)\pi}{6}\right) \right| \right) + \log \left(\left| \tan\left(\frac{(ik+1)\pi}{6}\right) \right| \right) \right]_0^\infty$$

$$\tan\left(\frac{(ik+2)\pi}{6}\right) = \frac{\tan\left(\frac{ik\pi}{6}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{ik\pi}{6}\right) \tan\left(\frac{\pi}{3}\right)} = \frac{i \operatorname{tanh}\left(\frac{k\pi}{6}\right) + \tan\frac{\pi}{3}}{1 - i \operatorname{tanh}\left(\frac{k\pi}{6}\right) \tan\frac{\pi}{3}}$$

$$\tan\left(\frac{(ik+1)\pi}{6}\right) = \frac{\tan\left(\frac{ik\pi}{6}\right) + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\left(\frac{ik\pi}{6}\right) \tan\left(\frac{\pi}{6}\right)} = \frac{i \operatorname{tanh}\left(\frac{k\pi}{6}\right) + \tan\frac{\pi}{6}}{1 - i \operatorname{tanh}\left(\frac{k\pi}{6}\right) \tan\frac{\pi}{6}}$$



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$$\lim_{n \rightarrow \infty} \tan\left(\frac{(ik+2)\pi}{6}\right) = \frac{\sqrt{3} + i}{1 - i\sqrt{3}} = \frac{1 - \sqrt{3}}{1 - \sqrt{3}} \cdot i = i$$

$$\lim_{n \rightarrow \infty} \tan\left(\frac{(ik+1)\pi}{6}\right) = \frac{\frac{1}{\sqrt{3}} + i}{1 - \frac{i}{\sqrt{3}}} = \frac{1 - \frac{i}{\sqrt{3}}}{1 - \frac{i}{\sqrt{3}}} \cdot i = i$$

$$\int_0^\infty R(k) dk = \frac{1}{i} \left(2 \log i - \log \left(\left| \tan \frac{\pi}{3} \right| \right) - \log \left(\left| \tan \frac{\pi}{6} \right| \right) \right) = \frac{2}{i} \cdot \log i = \frac{2}{i} \cdot \frac{\pi}{2} \cdot i = \pi$$

$$\int_0^\infty R(k) dk = \pi$$

1119. Find:

$$\Omega(n) = \int_0^{\frac{\pi}{2}} \frac{2(2n+1)x + \cos((4n+2)x)}{\sin x + \cos x} dx, n \in \mathbb{N}, n \geq 1$$

Proposed by Marin Chirciu-Romania

Solution by Rana Ranino-Setif-Algerie

Let $k = 2n + 1$, so $\Omega(k) = \int_0^{\frac{\pi}{2}} \frac{2k + \cos(2kx)}{\sin x + \cos x} dx$ and using $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\Omega(k) = \int_0^{\frac{\pi}{2}} \frac{2k\left(\frac{\pi}{2} - x\right) + \cos\left(2k\left(\frac{\pi}{2} - x\right)\right)}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{k\pi - 2k\pi + \cos(k\pi - 2kx)}{\sin x + \cos x} dx$$

$$2\Omega(k) = \int_0^{\frac{\pi}{2}} \frac{k\pi + \cos(2kx) + \cos(k\pi - 2kx)}{\sin x + \cos x} dx$$

$$\cos(2kx) + \cos(k\pi - 2kx) = 2\cos\left(\frac{k\pi}{2}\right) \cos(k\pi - 4kx) = 0$$

$$\Omega(k) = \frac{k\pi}{2} \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x} \stackrel{t=\tan\frac{x}{2}}{=} k\pi \int_0^1 \frac{dt}{1 + 2t - t^2} = k\pi \int_0^1 \frac{dt}{2 - (1-t)^2} =$$

$$= \frac{k\pi}{\sqrt{2}} \cdot \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)$$



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$$\text{Finally: } \Omega(n) = \frac{(2n+1)\pi}{\sqrt{2}} \cdot \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

1120. Prove that:

$$\int_0^1 \sqrt[6]{x^2 \left(1 - \sqrt[3]{x^2}\right)^3} \cdot \log x \cdot \log \left(1 - \sqrt[3]{x^2}\right) dx = \frac{1532}{375} - \frac{32}{25} \log 2 - \frac{3\pi^2}{10}$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} \Omega &= \int_0^1 \sqrt[6]{x^2 \left(1 - \sqrt[3]{x^2}\right)^3} \cdot \log x \cdot \log \left(1 - \sqrt[3]{x^2}\right) dx = \\ &= \left[\frac{d}{dv} \cdot \frac{d}{du} \int_0^1 x^{u-1} \left(1 - \sqrt[3]{x^2}\right)^{v-1} dx \right]_{(u,v)=\left(\frac{4}{3}, \frac{3}{2}\right)} \stackrel{t=\sqrt[3]{x^2}}{=} \\ &= \frac{3}{2} \left[\frac{d}{dv} \cdot \frac{d}{du} \int_0^1 t^{\frac{3u}{2}-1} (1-t)^{v-1} dt \right]_{(u,v)=\left(\frac{4}{3}, \frac{3}{2}\right)} = \frac{9}{4} \left[\frac{d}{dv} \cdot \frac{d}{du} \cdot \frac{\Gamma\left(\frac{3u}{2}\right) \Gamma(v)}{\Gamma\left(\frac{3u}{2} + v\right)} \right]_{(u,v)=\left(\frac{4}{3}, \frac{3}{2}\right)} = \\ &= \frac{9}{4} \cdot \frac{\Gamma\left(\frac{3}{2}\right) \Gamma(2)}{\Gamma\left(\frac{3}{2} + 2\right)} \left(-\Psi^{(1)}\left(\frac{3}{2} + 2\right) + \left(\Psi\left(\frac{3}{2}\right) - \Psi\left(\frac{3}{2} + 2\right)\right) \left(\Psi(2) - \Psi\left(\frac{3}{2} + 2\right)\right) \right) \end{aligned}$$

We know that:

$$\begin{aligned} \Psi^{(1)}(1+x) &= -\frac{1}{x^2} + \Psi^{(1)}(x); \\ \Psi^{(1)}\left(x + \frac{1}{2}\right) &= 4\Psi^{(1)}(2x) - \Psi^{(1)}(x) \\ \Omega &= \frac{9}{4} \cdot \frac{4}{15} \left(-\Psi^{(1)}\left(\frac{1}{2}\right) + \frac{4}{25} + \frac{4}{9} + 4 + \left(-\frac{2}{5} - \frac{2}{3}\right) \left(1 + \Psi(1) - \frac{2}{5} - \frac{2}{3} - 2 - \Psi\left(\frac{1}{2}\right)\right) \right) = \\ &= \frac{3}{5} \left(-\Psi^{(1)}\left(\frac{1}{2}\right) + \frac{1036}{225} + \frac{436}{225} - \frac{32}{15} \log 2 \right) \end{aligned}$$

But:



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$$\Psi^{(1)}\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^2} = 4 \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = 4 \left(\zeta(2) - \frac{1}{4} \zeta(2) \right) = 3\zeta(2) = \frac{\pi^2}{2}$$

Then:

$$\Omega = -\frac{3\pi^2}{5} + \frac{1}{5} - \frac{32}{5} \log$$

$$\int_0^1 \sqrt[6]{x^2 \left(1 - \sqrt[3]{x^2}\right)^3} \cdot \log x \cdot \log \left(1 - \sqrt[3]{x^2}\right) dx = \frac{1532}{375} - \frac{32}{25} \log 2 - \frac{3\pi^2}{10}$$

1121. Find a closed form:

$$\Omega = \int_0^1 \frac{x \cdot \log^2 x \cdot \log^2(1-x)}{\sqrt{1-x}} dx$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} \Omega &= \int_0^1 \frac{x \cdot \log^2 x \cdot \log^2(1-x)}{\sqrt{1-x}} dx = \left[\frac{\partial^2}{\partial \beta^2 \partial \alpha^2} \left(\frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \right) \right]_{(\alpha, \beta) = (2, \frac{1}{2})} \\ \frac{\partial^2}{\partial \alpha^2} \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta)} \right) &= \frac{\partial}{\partial \alpha} \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta)} (\Psi(\alpha) - \Psi(\alpha + \beta)) \right) = \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta)} (\Psi(\alpha) - \Psi(\alpha + \beta))^2 + \Psi^{(1)}(\alpha) - \Psi^{(1)}(\alpha + \beta) \\ \Omega &= \left[\frac{\partial^2}{\partial \beta^2} \left(\frac{\Gamma(2) \Gamma(\beta)}{\Gamma(\beta + 2)} (\Psi(2) - \Psi(\beta + 2))^2 + \Psi^{(1)}(2) - \Psi^{(1)}(\beta + 2) \right) \right]_{\beta = \frac{1}{2}} = \\ &= \left[\frac{\partial^2}{\partial \beta^2} \left(\frac{1}{\beta(1 + \beta)} (1 - \gamma + \Psi(\beta + 2))^2 + \zeta(2) - 1 - \Psi^{(1)}(\beta + 2) \right) \right]_{\beta = \frac{1}{2}} = \\ &= \left[\frac{\partial^2}{\partial \beta^2} \left(\frac{1}{(\frac{1}{2} + \beta)(\frac{3}{2} + \beta)} \left(1 - \gamma + \Psi\left(\beta + \frac{5}{2}\right) \right)^2 + \zeta(2) - 1 - \Psi^{(1)}\left(\beta + \frac{5}{2}\right) \right) \right]_{\beta=0} \end{aligned}$$

$$\text{Using: } f(\beta) = f(0) + f'(0)\beta + \frac{f''(0)}{2}\beta^2 = o(\beta^2)$$



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$$\begin{aligned}
 \frac{1}{\left(\frac{1}{2} + \beta\right)\left(\frac{3}{2} + \beta\right)} &= \frac{4}{3} - \frac{32}{9}\beta + \frac{208}{2}\beta^2 + o(\beta^2) \\
 \left(1 - \gamma + \Psi\left(\beta + \frac{5}{2}\right)\right)^2 &= \left(1 - \gamma + \Psi\left(\frac{5}{2}\right) - \Psi^{(1)}\left(\frac{5}{2}\right)\beta + 2\Psi^{(2)}\left(\frac{5}{2}\right)\beta^2 + o(\beta^2)\right)^2 = \\
 &= \left(1 - \gamma + \Psi\left(\frac{5}{2}\right)\right)^2 + \left(\frac{80}{9} - \pi^2\right)\left(1 - \gamma - \Psi^{(1)}\left(\frac{5}{2}\right)\beta\right) + \left(\frac{40}{9} + \frac{\pi^2}{2}\right)^2\beta^2 + o(\beta^2) \\
 \Psi^{(1)}\left(\beta + \frac{5}{2}\right) &= \Psi^{(1)}\left(\frac{5}{2}\right) + \Psi^{(2)}\left(\frac{5}{2}\right)\beta + \Psi^{(3)}\left(\frac{5}{2}\right)\beta^2 + o(\beta^2) \\
 \Psi\left(x + \frac{1}{2}\right) &= 2\Psi(2x) - \Psi(x) - 2\log 2 \\
 \Psi^{(1)}\left(x + \frac{1}{2}\right) &= 4\Psi^{(1)}(2x) - \Psi^{(1)}(x) \\
 \Psi^{(2)}\left(x + \frac{1}{2}\right) &= 6\Psi^{(2)}(2x) - \Psi^{(2)}(x) \\
 \Psi\left(\frac{1}{2}\right) &= -\gamma - 2\log 2, \Psi^{(1)}\left(\frac{1}{2}\right) = 3\zeta(2) = \frac{\pi^2}{2}, \Psi^{(2)}\left(\frac{1}{2}\right) = -14\zeta(3)
 \end{aligned}$$

We get:

$$\frac{\Omega 224}{3}\zeta(3)\log 2 - \frac{1456}{9}\zeta(3) + \frac{46592}{81} + \frac{1664}{27}\log^2 2 - \frac{8576}{27}\log 2 - \frac{2\pi^4}{3} + \frac{128}{9}\pi^2\log 2 - \frac{2336}{81}\pi^2$$

1122. In $\triangle ABC$ the following relationship holds:

$$\frac{1}{w_a} \int_{h_a}^{w_a} \frac{\sin x}{x} dx + \frac{1}{w_b} \int_{h_b}^{w_b} \frac{\sin x}{x} dx + \frac{1}{w_c} \int_{h_c}^{w_c} \frac{\sin x}{x} dx \leq \frac{R - 2r}{r(R + r)}$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution by Daniel Sitaru – Romania

$$\sum_{cyc} \frac{1}{w_a} \geq \frac{3}{R + r} \quad (\text{Milosevic} - 1990)$$

$$\sum_{cyc} \frac{1}{w_a} \int_{h_a}^{w_a} \frac{\sin x}{x} dx \leq \sum_{cyc} \frac{1}{w_a} \int_{h_a}^{w_a} \frac{1}{x} dx =$$



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$$\begin{aligned}
 &= \sum_{cyc} \frac{1}{w_a} (\log w_a - \log h_a) = \\
 &\stackrel{MVT \ (c_a \in (h_a, w_a))}{=} \sum_{cyc} \frac{1}{w_a} \cdot \frac{1}{c_a} (w_a - h_a) \leq \sum_{cyc} \frac{w_a - h_a}{w_a h_a} = \\
 &= \sum_{cyc} \frac{1}{h_a} - \sum_{cyc} \frac{1}{w_a} = \frac{1}{r} - \frac{3}{R+r} = \frac{R-2r}{r(R+r)}
 \end{aligned}$$

Equality holds for $a = b = c$

1123. If $a, b > 0$, $f(a, b) = \int_0^\infty \frac{(1-e^{-ax})^2 \cos(bx)}{x} dx$ then for $a, b > \frac{3}{2}$, prove

that:

$$f\left(\sqrt{ab} + \frac{a+b}{2}\right) + \log(a+b) > 1$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Dawid Bialek-Poland

$$\begin{aligned}
 f(a, b) &= \int_0^\infty \frac{(1-e^{-ax})^2 \cos(bx)}{x} dx \Rightarrow \frac{d}{da} f(a, b) = \int_0^\infty \frac{2x(e^{-ax} - e^{-2ax}) \cos(bx)}{x} dx \\
 &= 2 \int_0^\infty (e^{-ax} - e^{-2ax}) \cos(bx) dx
 \end{aligned}$$

The above integral is known as: $\int_0^\infty e^{-ax} \cos(bx) dx = \frac{a}{a^2+b^2}, a > 0$; (*)

So:

$$\begin{aligned}
 \frac{d}{da} f(a, b) &= 2 \int_0^\infty e^{-ax} \cos(bx) dx - 2 \int_0^\infty e^{-2ax} \cos(bx) dx \stackrel{(*)}{=} \\
 &2 \cdot \frac{a}{a^2+b^2} - 2 \cdot \frac{2a}{4a^2+b^2} = \frac{2a}{a^2+b^2} - \frac{4a}{4a^2+b^2} \\
 f(a, b) &= 2 \int \underbrace{\frac{a}{a^2+b^2}}_{u=a^2+b^2} da - 4 \int \underbrace{\frac{a}{4a^2+b^2}}_{z=4a^2+b^2} da = \int \frac{du}{u} - \frac{1}{2} \int \frac{dz}{z} =
 \end{aligned}$$



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$$= \log u - \frac{1}{2} \log z + C \stackrel{u=a^2+b^2}{=} \log(a^2 + b^2) - \frac{1}{2} \log(4a^2 + b^2) + C$$

$$f(\mathbf{0}, b) = 2 \int_0^\infty (e^{-0 \cdot x} - e^{-2 \cdot 0 \cdot x}) \cos(bx) dx = 0 \Rightarrow$$

$$f(\mathbf{0}, b) = \log(0^2 + b^2) - \frac{1}{2} \log(4 \cdot 0^2 + b^2) + C \Rightarrow C = -\log b$$

$$f(a, b) = \log(a^2 + b^2) - \frac{1}{2} \log(4a^2 + b^2) - \log b$$

$$f\left(\sqrt{ab} + \frac{a+b}{2}\right) + \log(a+b) > 1 \Rightarrow$$

$$\log\left(ab + \left(\frac{a+b}{2}\right)^2\right) - \frac{1}{2} \log\left(4ab + \left(\frac{a+b}{2}\right)^2\right) - \log\left(\frac{a+b}{2}\right) + \log(a+b) > 1$$

$$\log\left(ab + \left(\frac{a+b}{2}\right)^2\right) - \frac{1}{2} \log\left(4ab + \left(\frac{a+b}{2}\right)^2\right) + \log\left(\frac{2}{a+b} \cdot (a+b)\right) > 1$$

Let: $a = b > 0$ then

$$\log\left(a \cdot a + \left(\frac{a+a}{2}\right)^2\right) - \frac{1}{2} \log\left(4a \cdot a + \left(\frac{a+a}{2}\right)^2\right) + \log\left(\frac{2}{a+a} \cdot (a+a)\right) > 1$$

$$\log(2a^2) - \frac{1}{2} \log(5a^2) + \log 2 > 1 \Rightarrow \log\left(\frac{4\sqrt{5}}{5}a\right) > \log e \Rightarrow$$

$$\frac{4\sqrt{5}}{5}a > e$$

Note. $x \rightarrow \log x$ – increasing function.

$$a = b > \frac{\sqrt{5}e}{4} \dots > \frac{3}{2}, \text{ where } e = 2.718\dots$$

If the desired inequality holds for $a = b = \frac{3}{2} \Rightarrow$ inequality holds for $a > b > \frac{3}{2}$ and

$$b > a > \frac{3}{2}$$

1124. Prove without softs:

$$\left(\int_0^1 \frac{\sin^{-1}x}{\pi^2 - x^2} dx \right) \left(\int_0^1 \frac{\cos^{-1}x}{\pi^2 - x^2} dx \right) < \frac{1}{64}$$

Proposed by Hasan Mammadov-Azerbaijan

Solution by Daniel Sitaru – Romania

$$\begin{aligned} & \left(\int_0^1 \frac{\sin^{-1}x}{\pi^2 - x^2} dx \right) \left(\int_0^1 \frac{\cos^{-1}x}{\pi^2 - x^2} dx \right) \stackrel{AM-GM}{\leq} \\ & \leq \left(\frac{1}{2} \left(\int_0^1 \frac{\sin^{-1}x}{\pi^2 - x^2} dx + \int_0^1 \frac{\cos^{-1}x}{\pi^2 - x^2} dx \right) \right)^2 = \left(\frac{1}{2} \left(\int_0^1 \frac{\sin^{-1}x + \cos^{-1}x}{\pi^2 - x^2} dx \right) \right)^2 \\ & = \left(\frac{1}{2} \left(\int_0^1 \frac{\frac{\pi}{2}}{\pi^2 - x^2} dx \right) \right)^2 = \frac{\pi^2}{16} \left(\int_0^1 \frac{1}{\pi^2 - x^2} dx \right)^2 = \\ & = \frac{\pi^2}{16} \cdot \frac{1}{4\pi^2} \cdot \log^2 \left(\frac{\pi - 1}{\pi + 1} \right) = \frac{1}{64} \log^2 \left(\frac{\pi - 1}{\pi + 1} \right) < \\ & < \frac{1}{64} \log^2 e = \frac{1}{64} \end{aligned}$$

1125. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{|\sin x \cdot \sin(2x) \cdot \dots \cdot \sin(2019x)|}{\sin^{2019} x} dx \leq 2019! (b-a)$$

Poposed by Daniel Sitaru-Romania

Solution by Remus Florin Stanca-Romania

We prove that $\sin(kx) \leq k \sin x, \forall k \geq 1, k \in \mathbb{N}$ by using Mathematical Induction:

1. The statement $P(1): \sin x \leq \sin x$ is true.

2. We suppose that: $P(k): \sin(kx) \leq k \sin x$ is true

3. We prove that: $P(k+1): \sin((k+1)x) \leq (k+1) \sin x$ by using $P(k)$:

$$\sin((k+1)x) = \sin(kx+x) = \sin(kx)\cos x + \sin x \cos(kx); (1)$$



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$$\sin(kx) \leq ks\sin x \cdot \cos x \Rightarrow \sin(kx)\cos x \leq ks\sin x \cos x, x \in \left(0, \frac{\pi}{2}\right), \cos x > 0$$

$$\begin{aligned} \sin(kx)\cos x &\leq ks\sin x \cos x + (+\cos(kx)\sin x) \Rightarrow \\ \sin(kx)\cos x + \cos(kx)\sin x &\leq ks\sin x \cos x + \cos(kx)\sin x; (2) \end{aligned}$$

$$\cos x \leq 1 \cdot ks\sin x > 0 \left(x \in \left(0, \frac{\pi}{2}\right)\right) \Rightarrow ks\sin x \cos x \leq ks\sin x; (3)$$

$$\cos(kx) \geq 1 \cdot \sin x > 0 \Rightarrow \cos(kx)\sin x \leq \sin x; (4)$$

$$\xrightarrow{(3)+(4)} \sin(kx)\cos x + \cos(kx)\sin x \leq (k+1)\sin x \xrightarrow{(1)+(2)}$$

$$\sin((k+1)x) \leq (k+1)\sin x \Rightarrow \sin(kx) \leq ks\sin x, \forall k \geq 1, k \in \mathbb{N}; (*)$$

1. We know that: $\sin x \geq -\sin x$ true because $\sin x \geq 0, x \in \left(0, \frac{\pi}{2}\right)$

2. We suppose that $P(k): \sin(kx) \geq -ks\sin x$ is true

3. We suppose that $P(k+1): \sin((k+1)x) \geq -(k+1)\sin x$ is true by using $P(k)$:

$$\sin((k+1)x) = \sin(kx + x) = \sin(kx)\cos x + \cos(kx)\sin x; (5)$$

$$\sin(kx) \geq -ks\sin x \cdot \cos x > 0; \left(x \in \left(0, \frac{\pi}{2}\right)\right) \Rightarrow$$

$$\begin{aligned} \sin(kx)\cos x &\geq -ks\sin x \cos x + (+\cos(kx)\sin x) \Rightarrow \\ \sin(kx)\cos x + \cos(kx)\sin x &\geq -ks\sin x \cos x + \cos(kx)\sin x; (6) \end{aligned}$$

$$\cos x \leq 1 \cdot (-ks\sin x) \Rightarrow -ks\sin x \cos x \geq -ks\sin x; (7)$$

$$\cos(kx) + 1 = 2\cos^2\left(\frac{kx}{2}\right) \geq 0 \Rightarrow \cos(kx) + 1 \geq 0 \Rightarrow \cos(kx) \geq -1 \Rightarrow$$

$$\cos(kx)\sin x \geq -\sin x; (8)$$

$$(7) + (8) \Rightarrow -ks\sin x \cos x + \cos(kx)\sin x \geq -(k+1)\sin x$$

$$\xrightarrow{(5)+(6)} \sin((k+1)x) \geq -(k+1)\sin x; (\text{proved})$$

$$\sin(kx) \geq -ks\sin x, \forall k \geq 1, k \in \mathbb{N}; (**)$$

From $(*)$, $(**)$ we get: $-ks\sin x \leq \sin(kx) \leq ks\sin x \Rightarrow |\sin(kx)| \leq ks\sin x \Rightarrow$

$$\frac{|\sin(kx)|}{\sin x} \leq k \Rightarrow \prod_{k=1}^{2019} \frac{|\sin(kx)|}{\sin x} \leq \prod_{k=1}^n k \Rightarrow$$

$$\frac{|\sin x \cdot \sin(2) \cdot \dots \cdot \sin(2019x)|}{\sin^{2019} x} \leq 2019! \Rightarrow$$



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$$\int_a^b \frac{|\sin x \cdot \sin(2x) \cdots \sin(2019x)|}{\sin^{2019} x} dx \leq \int_a^b 2019! dx = 2019!$$

1126. If $x \in [-1, 1]$ then prove:

$$\frac{1}{\pi} \left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right| \leq 1, n \in \mathbb{N}, n \geq 1$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$\underbrace{\frac{1}{\pi} \left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right|}_I \leq 1, n \in \mathbb{N}, n \geq 1$$

We have:

$$A = \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt = \left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right| (\cos\theta + i\sin\theta)$$

$$\begin{cases} \cos\theta = \frac{\operatorname{Re}(A)}{I} \\ \sin\theta = \frac{\operatorname{Im}(A)}{I} \end{cases}$$

$$\left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right| = \left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n (\cos\theta - i\sin\theta) dt \right|$$

Since $I \in \mathbb{R}$, we have:

$$\left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right| = \left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n \cos\theta dt \right|; (1)$$

Since we have: $(xsint + i\sqrt{1-x^2})^n \cos\theta \leq |(xsint + i\sqrt{1-x^2})^n \cos\theta|$

On the other hand, since $|\cos\theta| \leq 1$, we have

$$|(xsint + i\sqrt{1-x^2})^n \cos\theta| = |(xsint + i\sqrt{1-x^2})^n| \cdot |\cos\theta| \leq \sqrt{1-x^2 \cos^2 t}$$

Thus, we have

$$\begin{aligned} \int_0^\pi (xsint + i\sqrt{1-x^2})^n \cos \theta dt &\leq \int_0^\pi |(xsint + i\sqrt{1-x^2})^n \cos \theta| dt \\ &\leq \int_0^\pi \sqrt{1-x^2 \cos^2 t} dt; (2) \end{aligned}$$

From(1),(2) we have:

$$\left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right| \leq \int_0^\pi \sqrt{1-x^2 \cos^2 t} dt$$

On the other hand, we have: $\sqrt{1-x^2 \cos^2 t} \leq 1, \forall x \in [-1, 1], t \in [0, \pi]$ so we have:

$$\begin{aligned} \left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right| &\leq \int_0^\pi 1 dt \Rightarrow \left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right| \leq \pi \Leftrightarrow \\ \frac{1}{\pi} \left| \int_0^\pi (xsint + i\sqrt{1-x^2})^n dt \right| &\leq 1, n \in \mathbb{N}, n \geq 1 \end{aligned}$$

1127. If $a, b \geq 0$ then:

$$\int_0^a \left(\int_0^t \log^5(x^2 + x + 2) dx \right) dt + \int_0^b \left(\int_0^t \log^5(x^2 + x + 2) dx \right) dt \geq ab \log 2$$

Proposed by Daniel Sitaru-Romania

Solution by Khaled Imouti-Damascus-Syria

Let be the function $f: (0, \infty) \rightarrow \mathbb{R}, f(t) = \log^5(t^2 + t + 2)$,

$$f'(t) = \frac{5(2t+1)}{t^2+t+2} \cdot \log^4(t^2+t+2)$$

$$\begin{aligned} f'(t) = 0 \Leftrightarrow t = -\frac{1}{2}; t^2 + t + 2 > 0 \Rightarrow f'(t) > 0, \forall t > 0 \Rightarrow f(t) > f(0), \forall t > 0 \\ \Rightarrow f(t) > \log^5 2 > \log 2; (\log 2 < 1), \forall t > 0 \end{aligned}$$

$$\int_0^a \left(\int_0^t \log^5(x^2 + x + 2) dx \right) dt + \int_0^b \left(\int_0^t \log^5(x^2 + x + 2) dx \right) dt \geq$$



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$$\begin{aligned} &\geq \int_0^a \log 2 \int_0^t dx + \int_0^b \log 2 \int_0^t dx = \frac{t^2}{2} \Big|_0^a \log 2 + \frac{t^2}{2} \Big|_0^b \log 2 \\ &= \frac{a^2 + b^2}{2} \cdot \log 2 \stackrel{AM-GM}{\geq} ab \log 2 \end{aligned}$$

1128. If $0 < a \leq b < 1$, $f: [0, 1] \rightarrow (0, \infty)$, f –continuous then:

$$\int_a^b \int_a^b \int_a^b \int_a^b (f(x) + f(z))(f(y) + f(t)) dx dy dz dt \geq 2(b-a)^3 \left(2 \int_a^b f(x) dx - 2b + 2a \right)$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$\begin{aligned} &\int_a^b \int_a^b (f(x) + f(z)) dx dz + \int_a^b \int_a^b (f(y) + f(t)) dy dt = \left(\int_a^b \int_a^b (f(x) + f(z)) dx dz \right)^2 = \\ &= \left(\int_a^b (f(x) + f(z))(b-a) dz \right)^2 = \left(\int_a^b (f(x) + f(x))(b-a) dx \right)^2 = \\ &= 4(b-a)^2 \left(\int_a^b f(x) dx \right)^2 \stackrel{(1)}{\geq} 2(b-a)^3 \left(2 \int_a^b f(x) dx - 2b + 2a \right) \\ (1) \Leftrightarrow &\left(\int_a^b f(x) dx \right)^2 \geq (b-a) \int_a^b f(x) dx - (b-a)^2 \Leftrightarrow \\ &\left(\int_a^b f(x) dx \right)^2 - (b-a) \int_a^b f(x) dx + (b-a)^2 \stackrel{(2)}{\geq} 0 \end{aligned}$$

But:

$$\left(\int_a^b f(x) dx \right)^2 - (b-a) \int_a^b f(x) dx + (b-a)^2 \geq$$

$$\geq \left(\int_a^b f(x) dx \right)^2 - 2(b-a) \int_a^b f(x) dx + (b-a)^2 = \left(\int_a^b f(x) dx - (b-a) \right)^2 \geq 0$$

$(0 < a \leq b < 1 \Rightarrow b-a \geq 0) \Rightarrow (2) \text{ true} \Rightarrow (1) \text{ true. Proved.}$

1129. $f: [0, 1] \rightarrow [0, \infty)$, f –twice derivable, $f'(x) > 0, f''(x) > 0, \forall x \in [0, 1]$

Prove that:

$$\int_0^1 f^3(x) dx + \frac{4}{27} \geq \left(\int_0^1 f(x) dx \right)^2$$

Proposed by Rajeev Rastogi-India

Solution 1 by Daniel Sitaru-Romania

$$\int_0^1 f^3(x) dx \stackrel{\text{Cebyshev}}{\geq} \frac{1}{(1-0)^2} \left(\int_0^1 f(x) dx \right)^3 \geq \left(\int_0^1 f(x) dx \right)^2 - \frac{4}{27} \quad (\text{to prove})$$

Denote:

$$\int_0^1 f(x) dx = a > 0$$

$$\begin{aligned} a^3 \geq a^2 - \frac{4}{27} &\Leftrightarrow 27a^3 + 4 \geq 27a^2 \Leftrightarrow 27a^2 \left(a - \frac{2}{3} \right) - 9a^2 + 4 \geq 0 \Leftrightarrow \\ &27a^2 \left(a - \frac{2}{3} \right) - 9a \left(a - \frac{2}{3} \right) - 6a + 4 \geq 0 \Leftrightarrow \\ &\left(a - \frac{2}{3} \right) (27a^2 - 9a - 6) \geq 0 \Leftrightarrow \left(a - \frac{2}{3} \right)^2 (27a + 9) \geq 0 \end{aligned}$$

Equality holds for:

$$\int_0^1 f(x) dx = a = \frac{2}{3}$$

Solution 2 by Abdallah El Farissi-Mostaganem-Algerie



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We have:

$$\begin{aligned}
 \int_0^1 \left(f(x) + \frac{1}{3} \right) \left(f(x) - \frac{2}{3} \right)^2 dx &\geq 0 \Rightarrow \int_0^1 \left(f(x) + \frac{1}{3} \right) \left(f^2(x) - \frac{4}{3}f(x) + \frac{4}{9} \right) dx = \\
 &= \int_0^1 \left(f(x) + \frac{1}{3} \right) \left[\left(f^2(x) - \frac{1}{3}f(x) + \frac{1}{9} \right) - \left(f(x) - \frac{1}{3} \right) \right] dx = \\
 &= \int_0^1 \left(f^3(x) + \frac{1}{27} \right) dx - \int_0^1 \left(f^2(x) - \frac{1}{9} \right) dx \geq 0 \Rightarrow \\
 &\int_0^1 \left(f^3(x) + \frac{1}{27} \right) dx \geq \int_0^1 \left(f^2(x) - \frac{1}{9} \right) dx \Rightarrow \\
 &\int_0^1 f^3(x) dx + \frac{4}{27} \geq \int_0^1 f^2(x) dx \stackrel{CBS}{\geq} \left(\int_0^1 f(x) \right)^2
 \end{aligned}$$

1130. If $a > 0$ then:

$$\int_{-a}^a \int_{-a}^a |(x+y)(1-xy)| dxdy \leq \frac{2}{9}(3a+a^3)^2$$

Proposed by Daniel Sitaru-Romania

Solution by George Florin Șerban-Romania

$$\begin{aligned}
 u \cdot v &\leq \frac{u^2 + v^2}{2}, \forall u, v \in \mathbb{R} \Leftrightarrow (u-v)^2 \geq 0, \forall u, v \in \mathbb{R} \\
 \int_{-a}^a \int_{-a}^a |(x+y)(1-xy)| dxdy &= \int_{-a}^a \int_{-a}^a |x+y| \cdot |1-xy| dxdy \leq \\
 &\leq \int_{-a}^a \int_{-a}^a \frac{|x+y|^2 + |1-xy|^2}{2} dxdy = \int_{-a}^a \int_{-a}^a \frac{x^2 + y^2 + 2xy + 1 - 2xy + x^2y^2}{2} dxdy = \\
 &= \int_{-a}^a \int_{-a}^a \frac{x^2 + y^2 + 1 + x^2y^2}{2} dxdy = \int_{-a}^a \int_{-a}^a \frac{(x^2 + 1)(y^2 + 1)}{2} dxdy =
 \end{aligned}$$



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$$\begin{aligned}
 &= \int_{-a}^a 2 \int_0^a \frac{x^2 + y^2 + 1 + x^2 y^2}{2} dx dy = \int_{-a}^a \left(\frac{a^3}{3} y^2 + \frac{a^3}{3} + a y^2 + a \right) dy = \\
 &= 2 \int_0^a \left(\frac{a^3}{3} y^2 + \frac{a^3}{3} + a y^2 + a \right) dy = 2 \left(\frac{a^6}{9} + \frac{2a^4}{3} + a^2 \right) = \frac{2}{9} (3a + a^3)^2
 \end{aligned}$$

1131. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \frac{(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y)}{(\sin^2 x \cdot \cos x + \tan y)^3} dx dy \geq (b-a)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\begin{aligned}
 &(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y) \stackrel{\text{Holder}}{\geq} \left(\sin^2 x \cdot \cos x + \sqrt[3]{\tan^2 y} \cdot \sqrt[3]{\tan y} \right)^3 \Leftrightarrow \\
 &(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y) \geq (\sin^2 x \cdot \cos x + \tan y)^3 \Leftrightarrow \\
 &\frac{(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y)}{(\sin^2 x \cdot \cos x + \tan y)^3} \geq 1 \Leftrightarrow \\
 &\int_a^b \int_a^b \frac{(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y)}{(\sin^2 x \cdot \cos x + \tan y)^3} dx dy \geq \int_a^b \int_a^b 1 dx dy = (b-a)^2
 \end{aligned}$$

Solution 2 by George Florin Șerban-Romania

$$\begin{aligned}
 &(\sin^2 x \cdot \cos x + \tan y)^3 = (\sin x \cdot \sin x \cdot \cos x + \sqrt[3]{\tan y} \cdot \sqrt[3]{\tan y} \cdot \sqrt[3]{\tan y})^3 \stackrel{\text{Holder}}{\leq} \\
 &\leq \left(\sin^3 x + (\sqrt[3]{\tan y})^3 \right) \left(\sin^3 x + (\sqrt[3]{\tan y})^3 \right) \left(\cos^3 x + (\sqrt[3]{\tan y})^3 \right) \Leftrightarrow \\
 &(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y) \geq (\sin^2 x \cdot \cos x + \tan y)^3 \Leftrightarrow \\
 &\frac{(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y)}{(\sin^2 x \cdot \cos x + \tan y)^3} \geq 1 \Leftrightarrow \\
 &\int_a^b \int_a^b \frac{(\sin^3 x + \tan y)^2 (\cos^3 x + \tan y)}{(\sin^2 x \cdot \cos x + \tan y)^3} dx dy \geq \int_a^b \int_a^b 1 dx dy = (b-a)^2
 \end{aligned}$$



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1132. $f: \mathbb{R} \rightarrow \mathbb{R}$, f – continuous, $a, b \in \mathbb{R}, a \leq b$. Prove that:

$$\int_a^b (f^8(x) + f^2(x)) dx + b - a \geq \int_a^b (f^5(x) + f(x)) dx$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$\int_a^b (f^8(x) + f^2(x)) dx + b - a \geq \int_a^b (f^5(x) + f(x)) dx \Leftrightarrow$$

$$\int_a^b (f^8(x) + f^2(x)) dx + \int_a^b f(x) dx \geq \int_a^b (f^5(x) + f(x)) dx \Leftrightarrow$$

$$\int_a^b (f^8(x) + f^2(x) + 1) dx \geq \int_a^b (f^5(x) + f(x)) dx \Leftrightarrow$$

$$(f^8(x) + f^2(x) + 1) \geq f^5(x) + f(x)$$

Let be the function: $g(y) = y^8 - y^5 + y^2 - y + 1$; ($y^8 \geq 0, y^2 \geq 0 \Rightarrow y^8 + y^2 \geq 2|y^5|$)

We must show that: $2|y^5| + 1 - y^5 - y \geq 0$

\therefore If $y < 0 \Rightarrow 2|y^5| + 1 - y^5 - y \geq 0$ is clearly true.

$$\therefore \text{If } y > 0 \Rightarrow 2|y^5| + 1 - y^5 - y = 2y^5 - y^5 - y + 1 = y^5 - y + 1$$

Let be the function: $h(y) = y^5 - y + 1$; $h'(y) = 5y^4 - 1$

$$h'(y) = 0 \Leftrightarrow y^4 = \frac{1}{5} \Rightarrow y^2 = \frac{1}{\sqrt{5}} \Rightarrow y_{1,2} = -\frac{1}{\sqrt[4]{5}} < 0 \text{ (contradiction with } y > 0)$$

and $y_{3,4} = \frac{1}{\sqrt[4]{5}}$

y	0	$\frac{1}{\sqrt[4]{5}}$	∞
$h'(y)$	$+ + + + + \mathbf{0} + + + + + + + + + + + + + + + + + + +$		
$h(y)$	$1 \nearrow \infty$		

Then $h(y) > 0$. So,

$$2|y^5| + 1 - y^5 - y \geq 0, \forall y \in \mathbb{R} \Rightarrow g(y) = y^8 - y^5 + y^2 - y + 1 > 0 \Rightarrow$$



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$$(f^8(x) + f^2(x) + 1) \geq f^5(x) + f(x) \Leftrightarrow$$

$$\int_a^b (f^8(x) + f^2(x)) dx + b - a \geq \int_a^b (f^5(x) + f(x)) dx$$

1133. Prove without softs:

$$\int_0^1 e^{x^2} dx \cdot \int_0^1 e^{-x^2} dx < \left(\frac{1+e}{2\sqrt{e}}\right)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$\Omega = \int_0^1 e^{x^2} dx \cdot \int_0^1 e^{-x^2} dx \stackrel{AM-GM}{\leq} \frac{1}{4} \left(\int_0^1 e^{x^2} dx + \int_0^1 e^{-x^2} dx \right)$$

$$I = \int_0^1 e^{x^2} dx \Rightarrow I^2 = \int_0^1 e^{x^2+y^2} dx dy; r^2 = x^2 + y^2, r dr d\theta = dx dy; (*)$$

$$I^2 = \int_0^{\sqrt{2}} \int_0^{\frac{\pi}{2}} e^{r^2} \cdot r dr d\theta = \int_0^{\sqrt{2}} \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} 2r \cdot e^{r^2} dr \right] d\theta = \frac{1}{2} \int_0^{\sqrt{2}} (e^2 - 1) d\theta = \frac{e^2 - 1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi(e^2 - 1)}{4} \Rightarrow I = \frac{\sqrt{\pi(e^2 - 1)}}{2} < \frac{\pi\sqrt{e}}{2}$$

$$J = \int_0^1 e^{-x^2} dx, J^2 = \int_0^1 e^{-(x^2+y^2)} dx dy \stackrel{(*)}{=} \int_0^{\sqrt{2}} \int_0^{\frac{\pi}{2}} e^{-r^2} \cdot r dr d\theta = \frac{\pi(1-e^{-2})}{4} \Rightarrow$$

$$J = \frac{\sqrt{\pi(1-e^{-2})}}{2} < \frac{\pi}{2} \sqrt{1 - \frac{1}{e^2}} \Rightarrow J < \frac{\sqrt{\pi}}{2}$$

$$\Omega = \int_0^1 e^{x^2} dx \cdot \int_0^1 e^{-x^2} dx < \frac{1}{4} \cdot \frac{\sqrt{\pi}}{2} (e+1) \stackrel{(1)}{<} \left(\frac{1+e}{2\sqrt{e}}\right)^2$$

$$(1) \Leftrightarrow \frac{\sqrt{\pi}}{2} < 1 + \frac{1}{e} \text{ (true)}$$



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Proved.

Solution 2 by Khaled Abd lmouti-Damascus-Syria

\therefore If $f: [a, b] \rightarrow [m, M]$, $m > 0$, f – continuous function then

$$\left(\int_a^b f(x) dx \right) \left(\int_a^b \frac{1}{f(x)} dx \right) \leq \frac{(m+M)^2}{4nM} \cdot (b-a)^2$$

Let be the function: $f: [0, 1] \rightarrow \mathbb{R}, f(x) = e^{x^2}; f'(x) = 2x \cdot e^{x^2} \geq 0$

$$f(\mathbf{0}) = \mathbf{1}, f(\mathbf{1}) = e$$

So,

$$\int_0^1 e^{x^2} dx \cdot \int_0^1 e^{-x^2} dx \leq \frac{(1+e)^2}{4e} \cdot (1-0) \Leftrightarrow$$

$$\int_0^1 e^{x^2} dx \cdot \int_0^1 e^{-x^2} dx < \left(\frac{1+e}{2\sqrt{e}} \right)^2$$

Solution 3 by Rovsen Pirguliev-Sumgait-Azerbaijan

From Chebyshev's integral inequality:

$$\left(\int_a^b f(x) dx \right) \cdot \left(\int_a^b g(x) dx \right) \leq (b-a) \cdot \int_a^b f(x)g(x) dx$$

We have:

$$Lhs = \int_0^1 e^{x^2} dx \cdot \int_0^1 e^{-x^2} dx \leq (\mathbf{1} - \mathbf{0}) \cdot \int_0^1 e^{x^2} \cdot e^{-x^2} dx = x|_0^1 = \mathbf{1}$$

It remains to prove that:

$$1 < \left(\frac{1+e}{2\sqrt{e}}\right)^2 \Leftrightarrow 1 < \frac{1+e}{2\sqrt{e}} \Leftrightarrow 2\sqrt{e} < 1+e \Leftrightarrow \sqrt{e} \stackrel{AM-GM}{\leq} \frac{e+1}{2} \text{ (true)}$$

Proved.



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1134. If $a, b, c > 0$ then:

$$\int_a^{2a} \int_b^{3b} \int_c^{4c} \left(\sqrt[6]{\frac{x+1}{y+1}} + \sqrt[8]{\frac{y+1}{z+1}} + \sqrt[10]{\frac{z+1}{x+1}} \right) dx dy dz \geq 15abc$$

Proposed by Daniel Sitaru-Romania

Solution by Michael Sterghiou-Greece

$$\int_a^{2a} \int_b^{3b} \int_c^{4c} \left(\sqrt[6]{\frac{x+1}{y+1}} + \sqrt[8]{\frac{y+1}{z+1}} + \sqrt[10]{\frac{z+1}{x+1}} \right) dx dy dz \geq 15abc; (1)$$

Let $\frac{x+1}{y+1} = u^2, \frac{y+1}{z+1} = v^2, \frac{z+1}{x+1} = t^2$ then $T = \sqrt[3]{u} + \sqrt[4]{v} + \sqrt[5]{t}$ with $uvt = 1$; (c); $u, v, t > 0$ as

$$x, y, z > 0$$

We will minimize $T(u, v, t)$ by $uvt = 1$.

Consider the Lagrangian $L(u, v, t, \lambda) = T(u, v, t) - \lambda(uvt - 1)$

For the extreme of T we need to look into the points that make the vector

$$\nabla L(u, v, t, \lambda) = \mathbf{0} \text{ or } \frac{\partial L}{\partial u} = \frac{\partial L}{\partial v} = \frac{\partial L}{\partial t} = \frac{\partial L}{\partial \lambda} = \mathbf{0} \text{ or}$$

$$\begin{cases} \frac{1}{3\sqrt[3]{u^2}} = \lambda vt = \lambda \cdot \frac{1}{u} \\ \frac{1}{4\sqrt[4]{v^3}} = \lambda ut = \lambda \cdot \frac{1}{v} \\ \frac{1}{5\sqrt[5]{t^4}} = \lambda uv = \lambda \cdot \frac{1}{t} \end{cases} \Rightarrow \begin{cases} u = 27\lambda^3; & (2) \\ v = 256\lambda^4; & (3) \\ t = 3125\lambda^5; & (4) \end{cases}, uvt = 1 \Rightarrow$$

$$uvt = 1 = 216 \cdot 10^5 \cdot \lambda^{12} \Rightarrow \lambda = \sqrt[12]{\frac{1}{216 \cdot 10^5}} \text{ as } \lambda > 0 \text{ (from (2) for example)}$$

Now, $(u_0, v_0, t_0) = (27\lambda^3, 256\lambda^4, 3125\lambda^5)$ and $T(u_0, v_0, t_0) = 12\lambda \cong 2.937 > \frac{5}{2}$

Hence $T > \frac{5}{2}$ and

$$\int_a^{2a} \int_b^{3b} \int_c^{4c} T dx dy dz \geq \int_a^{2a} \int_b^{3b} \int_c^{4c} \frac{5}{2} dx dy dz = \frac{5}{2} (4c - c)(3b - b)(2a - a) = 15abc.$$

Equality for $a = b = c = 0$. Done!



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1135. Find a closed form:

$$\Omega = \prod_{n=1}^{\infty} \log\left(2 + \frac{1}{n}\right) \log\left(2 - \frac{1}{n+1}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Samir HajAli-Damascus-Syria

$\prod_{n=1}^{\infty} (1 + b_n) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \log(1 + b_n) = -\infty \Leftrightarrow b_n < 1$ and $\sum_{n=1}^{\infty} b_n$ is diverge.

Now,

$$\begin{aligned} \Omega &= \prod_{n=1}^{\infty} \log\left(2 + \frac{1}{n}\right) \log\left(2 - \frac{1}{n+1}\right) = \\ &= \prod_{n=1}^{\infty} \left[1 + \log\left(2 + \frac{1}{n}\right) \log\left(2 - \frac{1}{n+1}\right) - 1 \right] \end{aligned}$$

$$\text{Where } b_n = \log\left(2 + \frac{1}{n}\right) \log\left(2 - \frac{1}{n+1}\right) - 1$$

$$\text{So, } \log\left(2 + \frac{1}{n}\right) \log\left(2 - \frac{1}{n+1}\right) \leq \log 3 \log\left(2 - \frac{1}{n+1}\right) < \log 2 \log 3 \approx 0.7,$$

$\forall n \in \mathbb{N}$. Therefore $b_n < 0, \forall n \in \mathbb{N}, n \geq 1$ and

$\sum_{n=1}^{\infty} \left[\log\left(2 + \frac{1}{n}\right) \log\left(2 - \frac{1}{n+1}\right) - 1 \right]$ is diverge, because

$$\lim_{n \rightarrow \infty} \left(\log\left(2 + \frac{1}{n}\right) \log\left(2 - \frac{1}{n+1}\right) - 1 \right) = \log^2 2 - 1 \neq 0 \Rightarrow$$

$b_n < 0$ and $\sum_{n=1}^{\infty} b_n$ is diverge.

Depending on theory we can conclude:

$$\Omega = \prod_{n=1}^{\infty} \log\left(2 + \frac{1}{n}\right) \log\left(2 - \frac{1}{n+1}\right) = 0$$

1136. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{n \cdot \arctan \frac{1}{n}}{n^2 + 1^2} + \frac{n \cdot \arctan \frac{2}{n}}{n^2 + 2^2} + \dots + \frac{n \cdot \arctan \frac{n}{n}}{n^2 + n^2} \right)$$

Proposed by Adil Abdullayev-Baku-Azerbaijan



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Solution by Daniel Sitaru – Romania

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n \cdot \arctan \frac{k}{n}}{n^2 + k^2} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n \cdot \arctan \frac{k}{n}}{n^2 \left(1 + \left(\frac{k}{n}\right)^2\right)} = \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\arctan \frac{k}{n}}{\left(1 + \left(\frac{k}{n}\right)^2\right)} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right), f: [0, 1] \rightarrow \mathbb{R}, f(x) = \frac{\arctan x}{1 + x^2} \\
 x_n^k &= \zeta_n^k = \frac{k}{n}, |\Delta_n| = \frac{1}{n} \rightarrow 0 \\
 \Omega &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(\frac{k}{n}\right) \left(\frac{k}{n} - \frac{k-1}{n}\right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(\zeta_n^k) (x_n^k - x_n^{k-1}) = \\
 &= \int_0^1 f(x) dx = \int_0^1 \frac{\arctan x}{1 + x^2} dx = \frac{1}{2} (\arctan^2 1 - \arctan^2 0) = \frac{\pi^2}{32}
 \end{aligned}$$

1137. Find a closed form:

$$\Omega = \sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot \left[\frac{n!}{e} \right], x \in [-1, 1), [*] - GIF$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Kamel Benaicha-Algiers-Algerie

$$\Omega = \sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot \left[\frac{n!}{e} \right] = \frac{1}{e} \cdot \frac{x}{1-x} + \sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot \left\{ \frac{n!}{e} \right\}, x \in [-1, 1]$$

We have:

$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \sum_{k=0}^n \frac{(-1)^k}{k!} + \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}$$

For $k \leq n: k! \mid n!$, then $n! \sum_{k=0}^n \frac{(-1)^k}{k!} \in \mathbb{Z}$

$$\left\{ \frac{n!}{e} \right\} = \left\{ n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right\} = \left\{ n! \sum_{p=0}^{\infty} \frac{(-1)^{n+1+p}}{(n+1+p)!} \right\} = \left\{ (-1)^{n+1} \sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \cdot \frac{n! p!}{(n+1+p)!} \right\}$$



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We know that:

$$\frac{n! p!}{(n+1+p)!} = \frac{\Gamma(n+1)\Gamma(p+1)}{\Gamma(n+1+p+1)} = \int_0^1 t^n (1-t)^p dt$$

$$\begin{aligned} \text{So: } \left\{ \frac{n!}{e} \right\} &= \left\{ (-1)^{n+1} \sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \cdot \int_0^1 t^n (1-t)^p dt \right\} = \\ &= \left\{ (-1)^{n+1} \int_0^1 t^n \sum_{p=0}^{\infty} \frac{(t-1)^p}{p!} dt \right\} = \left\{ \frac{(-1)^{n+1}}{e} \int_0^1 t^n e^t dt \right\} \\ (\because 0 \leq t \leq 1 \Rightarrow 0 \leq t^n \leq 1; 1 \leq e^t \leq e \Rightarrow t^n \leq t^n e^t \leq e t^n) \\ \left(\because 0 < \frac{t^n}{e} \leq \frac{1}{e} t^n e^t \leq t^n < 1 \right) \end{aligned}$$

$$\therefore \left\{ \frac{n!}{e} \right\} = \begin{cases} \frac{1}{e} \int_0^1 t^{2n+1} e^t dt, & \text{if } n \in 2\mathbb{N} + 1 \\ 1 - \frac{1}{e} \int_0^1 t^{2n} e^t dt, & \text{if } n \in 2\mathbb{N} \end{cases}$$

For $x \in [-1, 1]$, we have:

$$\begin{aligned} \Omega_1 &= \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \left(1 - \frac{1}{e} \int_0^1 t^{2n} e^t dt \right) + \frac{1}{e} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \left(\int_0^1 t^{2n+1} e^t dt \right) = \\ &= \cosh(x) - 1 - \frac{1}{e} \int_0^1 (\cosh(xt) - 1) e^t dt + \frac{1}{e} \int_0^1 \sinh(xt) e^t dt = \\ &= \cosh(x) - 1 - \frac{1}{e} \int_0^1 (\cosh(xt) - \sinh(xt)) e^t dt + \frac{1}{e} \int_0^1 e^t dt = \\ &= \cosh(x) - 1 - \frac{e^{1-x} - 1}{e(1-x)} + \frac{e-1}{e} \\ \Omega &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot \left[\frac{n!}{e} \right] = \frac{e^{1-x} - 1}{e(1-x)} - \cosh(x) \end{aligned}$$

Note. For $x = -1$ the series is not convergent.



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1138. Find a closed form:

$$\Omega = \prod_{n=1}^{\infty} \left(1 + \left(\frac{1}{\pi} \right)^{3^n} + \left(\frac{1}{\pi^2} \right)^{3^n} \right)$$

Proposed by Daniel Sitaru-Romania

Solution by Florentin Vișescu-Romania

$$\begin{aligned}
 \text{Let be } P &= (1 + a^3 + (a^3)^2)(1 + a^{3^2} + (a^{3^2})^2) \cdot \dots \cdot (1 + a^{3^n} + (a^{3^n})^2) \\
 (1 - a^3)P &= (1 - a^3)(1 + a^3 + (a^3)^2)(1 + a^{3^2} + (a^{3^2})^2) \cdot \dots \cdot (1 + a^{3^n} + (a^{3^n})^2) = \\
 &= (1 - a^{3^3})(1 + a^{3^2} + (a^{3^2})^2) \cdot \dots \cdot (1 + a^{3^n} + (a^{3^n})^2) = \\
 &\vdots \\
 &= (1 - a^{3^n})(1 + a^{3^n} + (a^{3^n})^2) = 1 - a^{3^{n+1}} \Rightarrow \\
 P &= \frac{1 - a^{3^{n+1}}}{1 - a^3} \\
 \Omega &= \lim_{n \rightarrow \infty} \prod_{n=1}^{\infty} \left(1 + \left(\frac{1}{\pi} \right)^{3^n} + \left(\frac{1}{\pi^2} \right)^{3^n} \right) = \lim_{n \rightarrow \infty} \frac{1 - \left(\frac{1}{\pi} \right)^{3^{n+1}}}{1 - \left(\frac{1}{\pi} \right)^3} = \frac{\pi^3}{\pi^3 - 1}
 \end{aligned}$$

1139. Find without any software:

$$\Omega = \lim_{(x,y) \rightarrow (0,0)} \left(\int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y} \sqrt{\tan x} dx \right) \left(\int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y} \sqrt{\cot x} dx \right)$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned}
 \Omega_1 &= \lim_{(x,y) \rightarrow (0,0)} \int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y} \sqrt{\tan x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\tan x} dx = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\sqrt{y}}{1+y^2} dy = 2 \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{u^2 du}{1+u^4} =
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \int_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}} \frac{dv}{1+v^4} \Rightarrow 2\Omega_1 = 2 \int_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}} \frac{\left(1+\frac{1}{u^2}\right)du}{u^2 + \frac{1}{u^2}} = 2 \int_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}} \frac{d\left(1-\frac{1}{u}\right)}{\left(u-\frac{1}{u}\right)^2 + 2} \\
 \Omega_1 &= \frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{1}{\sqrt{2}} \left(u - \frac{1}{u} \right) \right) \right]_{\frac{1}{\sqrt[4]{3}}}^{\sqrt[4]{3}} = \sqrt{2} \tan^{-1} \left(\frac{1}{\sqrt{2}} \left(\sqrt[4]{3} - \frac{1}{\sqrt[4]{3}} \right) \right) = \\
 &= \sqrt{2} \tan^{-1} \left(\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}} \right) \\
 \Omega_2 &= \lim_{(x,y) \rightarrow (0,0)} \int_{\frac{\pi}{6}+x}^{\frac{\pi}{3}-y} \sqrt{\cot x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\cot x} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\tan x} dx = \Omega_1 \\
 \Omega &= \Omega_1 \Omega_2 = 2 \left(\tan^{-1} \left(\frac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}} \right) \right)^2
 \end{aligned}$$

1140.

$$\Omega(n, r) = \sum_{k=0}^n \frac{(-1)^k}{3r+3k-2} \binom{n}{k}, r \in \mathbb{N}, r - \text{fixed}$$

$$\text{Find: } \omega(r) = \lim_{n \rightarrow \infty} \sqrt[n]{\Omega(n, r)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\begin{aligned}
 \Omega(n, r) &= \sum_{k=0}^n \frac{(-1)^k}{3r+3k-2} \binom{n}{k} = \sum_{k=0}^n \left(\int_0^1 x^{3r+3k-3} dx (-1)^k \binom{n}{k} \right) = \\
 &= \int_0^1 \sum_{k=0}^n x^{3r-3} \cdot x^{3k} \cdot (-1)^k \binom{n}{k} dx = \int_0^1 x^{3r-3} \cdot \sum_{k=0}^n x^{3k} \cdot (-1)^k \binom{n}{k} dx =
 \end{aligned}$$



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$$\begin{aligned}
 &= \int_0^1 x^{3r-3} \cdot (1-x^3)^n dx \stackrel{x^3=t}{=} \int_0^1 t^{r-1} \cdot (1-t)^n \cdot t^{-\frac{2}{3}} dt = \\
 &= \frac{1}{3} \int_0^1 t^{r-\frac{2}{3}-1} \cdot (1-t)^{n+1-1} dt = \frac{1}{3} B\left(r - \frac{2}{3}, n + 1\right) = \frac{1}{3} \cdot \frac{\Gamma\left(r - \frac{2}{3}\right) \Gamma(n + 1)}{\Gamma\left(r + n + \frac{1}{3}\right)} \\
 \omega(r) &= \lim_{n \rightarrow \infty} \sqrt[n]{\Omega(n, r)} \stackrel{C-D'A}{=} \lim_{n \rightarrow \infty} \frac{\Omega(n + 1, r)}{\Omega(n, r)} = \\
 &= \lim_{n \rightarrow \infty} \frac{\Gamma\left(r - \frac{2}{3}\right) \Gamma(n + 2)}{\Gamma\left(r + n + \frac{4}{3}\right)} \cdot \frac{\Gamma\left(r + n + \frac{1}{3}\right)}{\Gamma\left(r - \frac{2}{3}\right) \Gamma(n + 1)} = \\
 &= \lim_{n \rightarrow \infty} \frac{(n + 1)! \Gamma\left(r + n + \frac{1}{3}\right)}{\left(r + n + \frac{1}{3}\right) \Gamma\left(r + n + \frac{1}{3}\right) \cdot n!} = \lim_{n \rightarrow \infty} \frac{n + 1}{r + n + \frac{1}{3}} = 1
 \end{aligned}$$

Solution 2 by Samir HajAli-Damascus-Syria

$$\begin{aligned}
 \Omega(n, r) &= \sum_{k=0}^n \frac{(-1)^k}{3r + 3k - 2} \binom{n}{k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \int_0^1 x^{3r+3k-3} dx = \\
 &= \int_0^1 \sum_{k=0}^n (-1)^k \binom{n}{k} x^{3r-3} \cdot x^{3k} dx = \int_0^1 x^{3r-3} \cdot \sum_{k=0}^n (-1)^k \binom{n}{k} x^{3k} dx = \\
 &= \int_0^1 (x^3)^{r-1} \cdot (1-x^3)^n dx \stackrel{x^3=t}{=} \int_0^1 t^{r-1} \cdot (1-t)^n \cdot t^{-\frac{2}{3}} dt = \\
 &= \frac{1}{3} \int_0^1 t^{r-\frac{2}{3}-1} \cdot (1-t)^{n+1-1} dt = \frac{1}{3} B\left(r - \frac{2}{3}, n + 1\right) \\
 \omega(r) &= \lim_{n \rightarrow \infty} \sqrt[n]{\Omega(n, r)} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{3} \cdot \frac{\Gamma\left(r - \frac{2}{3}\right) \Gamma(n + 1)}{\Gamma\left(r + n + \frac{1}{3}\right)}} =
 \end{aligned}$$



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$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{\Gamma(r - \frac{2}{3})}{3} \cdot \frac{\Gamma(n+1)}{\Gamma(r+n+\frac{1}{3})}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{\Gamma(n+1)}{\Gamma(r+n+\frac{1}{3})}} \stackrel{C-D'A}{=} \\
 &= \lim_{n \rightarrow \infty} \frac{\Gamma(n+2)}{\Gamma(r+n+\frac{4}{3})} \cdot \frac{\Gamma(r+n+\frac{1}{3})}{\Gamma(r-\frac{2}{3}) \Gamma(n+1)} = \lim_{n \rightarrow \infty} \frac{n+1}{r+n+\frac{1}{3}} = 1
 \end{aligned}$$

Solution 3 by Toby Joshua-Nigeria

$$\begin{aligned}
 \Omega(n, r) &= \sum_{k=0}^n \frac{(-1)^k}{3r+3k-2} \binom{n}{k} = \sum_{k=0}^n (-1)^k \int_0^1 y^{3r+3k-3} dy \binom{n}{k} = \\
 &= \int_0^1 y^{3r-3} dy \sum_{k=0}^n (-y^3)^k \binom{n}{k} = \int_0^1 y^{3r-3} \cdot (1-y^3)^n dy = \\
 &= \frac{1}{3} \int_0^1 y^{(r-\frac{2}{3})-1} \cdot (1-y^3)^{(n+1)-1} dy = \frac{1}{3} B\left(r-\frac{2}{3}, n+1\right) = \\
 &= \frac{1}{3} B\left(r-\frac{2}{3}, n+1\right) = \frac{1}{3} \cdot \frac{\Gamma(r-\frac{2}{3}) \Gamma(n+1)}{\Gamma(r+n+\frac{1}{3})}; \quad (1) \\
 \omega(r) &= \lim_{n \rightarrow \infty} \sqrt[n]{\Omega(n, r)} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{3} \cdot \frac{\Gamma(r-\frac{2}{3}) \Gamma(n+1)}{\Gamma(r+n+\frac{1}{3})}} = \\
 &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{\Gamma(r-\frac{2}{3})}{3} \cdot \frac{\Gamma(n+1)}{\Gamma(r+n+\frac{1}{3})}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{\Gamma(n+1)}{\Gamma(r+n+\frac{1}{3})}} = \lim_{n \rightarrow \infty} \left(\frac{n!}{(r+n-\frac{2}{3})!} \right)^{\frac{1}{n}}
 \end{aligned}$$

Considering the asymptotic expansion of $n!$ and $(n+x)!$

$$\omega(r) \sim \lim_{n \rightarrow \infty} \frac{\frac{n}{e}}{r+n-\frac{2}{3}} = \lim_{n \rightarrow \infty} \frac{n}{r+n-\frac{2}{3}} = \lim_{n \rightarrow \infty} \frac{1}{\frac{r}{n} + 1 - \frac{2}{3n}} = 1$$

1141. Find:

$$\Omega = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin^{-1} \frac{1}{\sqrt{n^2 + 2k-1}} + \sin \frac{k\pi}{n} \sin \frac{k\pi}{n^2} \right)$$

Proposed by Rajeev Rastogi-India



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Solution 1 by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin^{-1} \frac{1}{\sqrt{n^2 + 2k-1}} + \sin \frac{k\pi}{n} \sin \frac{k\pi}{n^2} \right) = \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin^{-1} \frac{1}{\sqrt{n^2 + 2k-1}} + \frac{k\pi}{n^2} \sin \frac{k\pi}{n} \right) \\
 &\quad \frac{1}{\sqrt{n^2 + 2k-1}} = \frac{1}{n} \left(1 + \frac{2k-1}{n^2} \right)^{-\frac{1}{2}} \sim \frac{1}{n} \left(1 - \frac{2k-1}{2n^2} \right) \\
 \Omega &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(1 - \frac{2k-1}{2n^2} \right) + \pi \int_0^1 x \sin(\pi x) dx = \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(n - \frac{n(n+1)}{2n^2} + \frac{1}{n} \right) + 1 + \pi \int_0^1 \cos(\pi x) dx = 2 \\
 \Omega &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin^{-1} \frac{1}{\sqrt{n^2 + 2k-1}} + \sin \frac{k\pi}{n} \sin \frac{k\pi}{n^2} \right) = 2
 \end{aligned}$$

Solution 2 by Florică Anastase-Romania

$$\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1 \Rightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ such that } \forall n \geq n_\varepsilon \text{ we get:}$$

$$1 - \varepsilon \leq \frac{\sin^{-1} x}{x} \leq 1 + \varepsilon \Leftrightarrow (1 - \varepsilon)x \leq \sin^{-1} x \leq (1 + \varepsilon)x$$

So, we have:

$$(1 + \varepsilon) \frac{1}{\sqrt{n^2 + 2k-1}} \leq \sin^{-1} \frac{1}{\sqrt{n^2 + 2k-1}} \leq (1 + \varepsilon) \frac{1}{\sqrt{n^2 + 2k-1}}$$

$$(1 - \varepsilon) \sum_{k=1}^n \frac{1}{\sqrt{n^2 + 2k-1}} \leq \sum_{k=1}^n \sin^{-1} \frac{1}{\sqrt{n^2 + 2k-1}} \leq (1 + \varepsilon) \sum_{k=1}^n \frac{1}{\sqrt{n^2 + 2k-1}}$$

$$\sum_{k=1}^n \frac{1}{\sqrt{n^2 + 2k-1}} = \frac{1}{n} \cdot \sum_{k=1}^n \frac{1}{\sqrt{1 + \frac{2k-1}{n^2}}} \cong \frac{1}{n} \cdot \sum_{k=1}^n \left(1 - \frac{2k-1}{2n^2} \right) =$$



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$$= \frac{1}{n} \cdot \left(n - \frac{n(n+1)-1}{2n^2} \right)$$

$$\begin{aligned}\Omega &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin^{-1} \frac{1}{\sqrt{n^2 + 2k-1}} + \sin \frac{k\pi}{n} \sin \frac{k\pi}{n^2} \right) = \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{\sqrt{n^2 + 2k-1}} + \sin \frac{k\pi}{n} \cdot \frac{\sin \frac{k\pi}{n^2}}{\frac{k\pi}{n^2}} \cdot \frac{k\pi}{n^2} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \left(n - \frac{n(n+1)-1}{2n^2} \right) + \sum_{k=1}^n \left(\frac{k\pi}{n^2} \cdot \sin \frac{k\pi}{n} \right) \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \left(n - \frac{n(n+1)-1}{2n^2} \right) + \frac{1}{n} \cdot \sum_{k=1}^n \left(\frac{k\pi}{n} \cdot \sin \frac{k\pi}{n} \right) \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \left(n - \frac{n(n+1)-1}{2n^2} \right) + \pi \int_0^1 x \sin(\pi x) dx \right) = \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(n - \frac{n(n+1)}{2n^2} + \frac{1}{n} \right) + 1 + \pi \int_0^1 \cos(\pi x) dx = 2 \\ \Omega &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sin^{-1} \frac{1}{\sqrt{n^2 + 2k-1}} + \sin \frac{k\pi}{n} \sin \frac{k\pi}{n^2} \right) = 2\end{aligned}$$

1142. Find:

$$\Omega = \cos^{12} 1^\circ + \cos^{12} 2^\circ + \cos^{12} 3^\circ + \dots + \cos^{12} 89^\circ$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Asmat Quatea-Kabul-Afghanistan

$$\begin{aligned}\Omega &= \cos^{12} 1^\circ + \cos^{12} 2^\circ + \cos^{12} 3^\circ + \dots + \cos^{12} 89^\circ = \\ &= (\cos^{12} 1^\circ + \sin^{12} 1^\circ) + (\cos^{12} 2^\circ + \sin^{12} 2^\circ) + \dots + (\cos^{12} 44^\circ + \sin^{12} 44^\circ) + \cos^{12} 45^\circ \\ (\because a^6 + b^6 &= (a+b)^6 - 6ab(a+b)^4 + 9a^2b^2(a+b)^2 - 2a^3b^3) \\ \left\{ \begin{array}{l} a = \sin^2 x \\ b = \cos^2 x \end{array} \right. &\Rightarrow \left\{ \begin{array}{l} a+b = 1 \\ a \cdot b = \frac{\sin^2 2x}{4} = \frac{1 - \cos 4x}{8} \end{array} \right.\end{aligned}$$



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$$\begin{aligned}
 \sin^{12}x + \cos^{12}x &= 1 - 6\left(\frac{1 - \cos 4x}{8}\right) + 9\left(\frac{1 - \cos 4x}{8}\right)^2 - 2\left(\frac{1 - \cos 4x}{8}\right)^3 = \\
 &= \frac{123 \cdot \cos 4x + 33 \cdot \cos^2 4x + \cos^3 4x + 99}{256} = \\
 &= \frac{492 \cdot \cos 4x + 132 \cdot \cos^2 4x + 4 \cdot \cos^3 4x + 396}{1024} = \\
 &= \frac{492 \cdot \cos 4x + 132 \cdot \cos^2 4x + \cos 12x + 3 \cdot \cos 4x + 396}{1024} = \\
 &= \frac{495 \cdot \cos 4x + 132 \cdot \cos^2 4x + \cos 12x + 396}{1024} = \\
 &= \frac{495 \cdot \cos 4x + 66 \cdot \cos 8x + 66 + \cos 12x + 396}{1024} = \\
 \sin^{12}x + \cos^{12}x &= \frac{\cos 12x + 66 \cdot \cos 8x + 495 \cdot \cos 4x + 462}{2^{10}} \\
 S(x, n) &= \sum_{k=1}^n \cos kx = \cos\left(\frac{(n+1)x}{2}\right) \cdot \frac{\sin\left(\frac{nx}{2}\right)}{\sin\frac{x}{2}} \\
 \sum_{x=1}^{44} (\sin^{12}x + \cos^{12}x) &= \sum_{x=1}^{44} \frac{\cos 12x + 66 \cdot \cos 8x + 495 \cdot \cos 4x + 462}{2^{10}} = \\
 &= \frac{S(12, 44) + 66 \cdot S(8, 44) + 495 \cdot S(4, 44) + 462 \cdot 44}{1024} \\
 \therefore S(x, n) &= \cos\left(\frac{(n+1)x}{2}\right) \cdot \frac{\sin\left(\frac{nx}{2}\right)}{\sin\frac{x}{2}} \\
 S(8, 44) &= \cos\left(\frac{45 \cdot 8}{2}\right) \cdot \frac{\sin\left(\frac{44 \cdot 8}{2}\right)}{\sin 4} = -1 \\
 S(12, 44) &= \cos\left(\frac{45 \cdot 12}{2}\right) \cdot \frac{\sin\left(\frac{44 \cdot 12}{2}\right)}{\sin 6} = 0 \\
 S(4, 44) &= \cos\left(\frac{45 \cdot 4}{2}\right) \cdot \frac{\sin\left(\frac{44 \cdot 4}{2}\right)}{\sin 2} = 0 \\
 \sum_{x=1}^{44} (\sin^{12}x + \cos^{12}x) &= \frac{0 + 66 \cdot (-1) + 495 \cdot 0 + 462 \cdot 44}{1024} = \frac{20262}{1024}
 \end{aligned}$$



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$$\frac{20262}{1024} + \cos^{12} 45^\circ = \frac{20262}{1024} + \frac{1}{64} = \frac{10139}{512}$$

$$\text{So, } \Omega = \cos^{12} 1^\circ + \cos^{12} 2^\circ + \cos^{12} 3^\circ + \dots + \cos^{12} 89^\circ = \frac{10139}{512}$$

1143. If

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{(\pi k)^2} + \frac{1}{(\pi k)^6} \right) = \sin a \cdot \sin b \cdot \sin h c \text{ then we have:}$$

$$a^4 + b^4 + c^4 = 1$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Benaicha-Algiers-Algerie

$$\Omega = \prod_{k=1}^{\infty} \left(1 + \frac{1}{(\pi k)^2} + \frac{1}{(\pi k)^6} \right)$$

$$\text{Put: } x = \frac{1}{(\pi k)^2} \Rightarrow \Omega = \prod_{k=1}^{\infty} (x^3 + x + 1)$$

$$x^3 + x + 1 = (x - \alpha)(x - \beta)(x - \gamma) = -\alpha\beta\gamma \left(1 - \frac{x}{\alpha}\right) \left(1 - \frac{x}{\beta}\right) \left(1 - \frac{x}{\gamma}\right) =$$

$$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$$

$$\therefore (A): \begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha\beta + \beta\gamma + \gamma\alpha = 1 \\ \alpha\beta\gamma = -1 \end{cases}$$

$$\therefore x^3 + x + 1 = \left(1 - \frac{x}{\alpha}\right) \left(1 - \frac{x}{\beta}\right) \left(1 - \frac{x}{\gamma}\right)$$

$$\text{So, } \Omega = \prod_{k=1}^{\infty} \left(1 - \frac{1}{\alpha(\pi k)^2}\right) \left(1 - \frac{1}{\beta(\pi k)^2}\right) \left(1 - \frac{1}{\gamma(\pi k)^2}\right)$$

We know that:

$$\prod_{k=1}^{\infty} \left(1 - \left(\frac{x}{k}\right)^2\right) = \frac{\sin \pi x}{\pi x}$$

$$\Omega = \sqrt{\alpha} \sin \frac{1}{\sqrt{\alpha}} \cdot \sqrt{\beta} \sin \frac{1}{\sqrt{\beta}} \cdot \sqrt{\gamma} \sin \frac{1}{\sqrt{\gamma}} = \sqrt{\alpha\beta\gamma} \cdot \sin \frac{1}{\sqrt{\alpha}} \cdot \sin \frac{1}{\sqrt{\beta}} \cdot \sin \frac{1}{\sqrt{\gamma}} \stackrel{\alpha\beta\gamma=-1}{=} \quad$$

$$\stackrel{\alpha\beta\gamma=-1}{=} i \cdot \sin \frac{1}{\sqrt{\alpha}} \cdot \sin \frac{1}{\sqrt{\beta}} \cdot \sin \frac{1}{\sqrt{\gamma}} = \sin \frac{1}{\sqrt{\alpha}} \cdot \sin \frac{1}{\sqrt{\beta}} \cdot \sinh \frac{i}{\sqrt{\gamma}}$$



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So: $a = \frac{1}{\sqrt{\alpha}}$; $b = \frac{1}{\sqrt{\beta}}$; $c = \frac{1}{\sqrt{\gamma}}$ where α, β, γ are solutions of $x^3 + x + 1 = 0$

$$a^4 + b^4 + c^4 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2}{(\alpha\beta\gamma)^2} \stackrel{\alpha\beta\gamma=-1}{=} (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$$

$$\text{Or: } \begin{cases} \alpha^3 + \alpha + 1 = 0; & (1) \\ \beta^3 + \beta + 1 = 0; & (2) \\ \gamma^3 + \gamma + 1 = 0; & (3) \end{cases}$$

$$\alpha(\alpha^3 + \alpha + 1) + \beta(\beta^3 + \beta + 1) + \gamma(\gamma^3 + \gamma + 1) = 0 \Leftrightarrow$$

$$\alpha^4 + \beta^4 + \gamma^4 + \alpha^2 + \beta^2 + \gamma^2 + \alpha + \beta + \gamma = 0; \quad (E)$$

$$(E) \Leftrightarrow \alpha^4 + \beta^4 + \gamma^4 + \alpha^2 + \beta^2 + \gamma^2 = 0; \quad (\alpha + \beta + \gamma = 0) \Leftrightarrow$$

$$(\alpha^2 + \beta^2 + \gamma^2)^2 - 2((\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2) + (\alpha^2 + \beta^2 + \gamma^2) = 0 \Leftrightarrow$$

$$\left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \right)^2 - 2((\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2)$$

$$+ \left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \right) = 0$$

$$2(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - ((\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2) - (\alpha\beta + \beta\gamma + \gamma\alpha) = 0$$

$$2 - ((\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2) - 1 = 0; \quad (\alpha\beta + \beta\gamma + \gamma\alpha = 1) \text{ from (A)}$$

$$(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = 1 \Rightarrow a^4 + b^4 + c^4 = 1$$

1144. Find without any software:

$$\Omega = \frac{1}{1 + \tan 6^\circ} + \frac{1}{1 - \tan 54^\circ} + \frac{1}{1 + \tan 66^\circ}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Santos Martins Junior-Brussels-Belgium

Let $a = \tan 6^\circ$, $b = \tan 54^\circ$, $c = \tan 66^\circ$

We have to compute: $\Omega = \frac{1}{1+a} + \frac{1}{1-b} + \frac{1}{1+c}$

$$1) \tan 60^\circ = \tan(66^\circ - 6^\circ) = \frac{c-a}{1+ac} \Leftrightarrow \frac{c-a}{1+ac} = \sqrt{3}; \quad (1)$$

$$2) \tan 60^\circ = \tan(54^\circ + 6^\circ) = \frac{a+b}{1-bc} \Leftrightarrow \frac{a+b}{1-bc} = \sqrt{3}; \quad (2)$$

$$3) \tan 60^\circ = -\tan 120^\circ = -\tan(54^\circ + 66^\circ) = -\frac{b+c}{1-bc} = \frac{b+c}{bc-1} \Rightarrow \frac{b+c}{bc-1} = \sqrt{3}; \quad (3)$$



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Since $\frac{c-a}{1+ac} = \sqrt{3} = \frac{a+b}{1-bc}$ using the componendo rule, we have:

$$\frac{(c-a) + (a+b)}{(1+ac) + (1-ab)} = \sqrt{3} \Leftrightarrow \frac{c+b}{2+ac-ab} = \sqrt{3}; \quad (4)$$

From (3),(4) we have:

$$\frac{b+c}{bc-1} = \frac{c+b}{2+ac-ab} \Leftrightarrow ab+bc-ca=3; \quad (5)$$

We also know the following identity:

$$\begin{aligned} & [(1 - (\tan p \cdot \tan q + \tan q \cdot \tan r + \tan r \cdot \tan p)) \cdot \tan(p+q+r)] \\ & = \tan p + \tan q + \tan r - \tan p \cdot \tan q \cdot \tan r; \end{aligned} \quad (6)$$

Applying here for $p = 6^\circ, q = 54^\circ, r = 66^\circ$, and with

$\tan(p+q+r) = \tan 126^\circ = -\tan 54^\circ = -b$; (7) then (6) becomes as:

$$[1 - (ab + bc + ca)] \cdot (-b) = (a + b + c) - abc; \quad (8)$$

From (5),(8) we get:

$$\begin{aligned} & [1 - (3 + 2ca)] \cdot (-b) = (a + b + c) - abc \Leftrightarrow \\ & -b + b \cdot (3 + 2ca) = a + b + c - abc \Leftrightarrow 3abc = a + c - b; \end{aligned} \quad (9)$$

For $a = \tan 6^\circ, b = \tan 54^\circ, c = \tan 66^\circ$ we get:

$$ab + bc - ca = 3 \text{ and } a + c - b = 3abc.$$

$$\begin{aligned} \Omega &= \frac{1}{1+a} + \frac{1}{1-b} + \frac{1}{1+c} = \frac{(1-b)(1+c) + (1+a)(1+c) + (1+a)(1-b)}{(1+a)(1-b)(1+c)} = \\ &= \frac{3 + 2(a+c-b) - (ab+bc-ca)}{1-(ab+bc-ca)+a+c-b-abc} = \frac{3 + 6abc - 3}{1 - 3 + 3abc - abc} = \\ &= \frac{3abc}{abc-1} \end{aligned}$$

$$\text{Also: } a = \tan 6^\circ = \tan(36^\circ - 30^\circ) = \frac{\tan 36^\circ - \tan 30^\circ}{1 + \tan 36^\circ \tan 30^\circ}$$

$$b = \tan 54^\circ = \cot 36^\circ = \frac{1}{\tan 36^\circ}$$

$$c = \tan 66^\circ = \tan(36^\circ + 30^\circ) = \frac{\tan 36^\circ + \tan 30^\circ}{1 - \tan 36^\circ \tan 30^\circ}$$

$$abc = \frac{\tan^2 36^\circ - \tan^2 30^\circ}{1 - \tan^2 36^\circ \cdot \tan^2 30^\circ} \cdot \frac{1}{\tan 36^\circ}$$



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$$\left(\therefore \tan 30^\circ = \frac{\sqrt{3}}{3}; \tan 36^\circ = \sqrt{5 - 2\sqrt{5}} \right)$$

$$abc = \frac{3}{\sqrt{3}} \cdot \frac{7 - 3\sqrt{5}}{\sqrt{5} - 1}$$

$$\Omega = \frac{3abc}{abc - 1}$$

1145. For $a, n \geq 1$. Find:

$$\Omega(a) = \int_1^{2n} \frac{\log(ax)}{x^2 + (2n+1)x + 2n} dx$$

Proposed by Marin Chirciu-Romania

Solution 1 by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} \Omega(a) &= \int_1^{2n} \frac{\log(ax)}{x^2 + (2n+1)x + 2n} dx = \frac{1}{2n-1} \left(\int_1^{2n} \frac{\log(ax)}{1+x} dx - \int_1^{2n} \frac{\log(ax)}{2n+x} dx \right) \\ &\int_1^{2n} \frac{\log x}{\alpha+x} dx = - \int_{\frac{1}{2n}}^1 \frac{\log t}{t(1+at)} dt = - \int_{\frac{1}{2n}}^1 \left(\frac{\log t}{t} - \frac{\alpha \log t}{1+at} \right) dt = \\ &= \frac{1}{2} \log^2(2n) + \log(2n) \log\left(1 + \frac{\alpha}{2n}\right) - \int_{\frac{1}{2n}}^1 \frac{\log(1+at)}{t} dt = \\ &= \frac{1}{2} \log^2(2n) + \log(2n) \log\left(1 + \frac{\alpha}{2n}\right) + \text{Li}_2(-\alpha) - \text{Li}_2\left(-\frac{\alpha}{2n}\right) \\ \Omega(a, n) &= \frac{1}{2n-1} \left(\log(2n) \left(\log\left(1 + \frac{1}{2n}\right) - \log 2 \right) + \text{Li}_2(-1) - \text{Li}_2\left(-\frac{1}{2n}\right) \right. \\ &\quad \left. - \text{Li}_2(-2n) + \text{Li}_2(-1) + \log(a) \log\left(\frac{(2n+1)^2}{8n}\right) \right) = \\ &= \frac{1}{2n-1} \left(\log(2n) \log\left(\frac{1+2n}{4n}\right) - \text{Li}_2\left(-\frac{1}{2n}\right) - \text{Li}_2(-2n) + \log a \log\left(\frac{(2n+1)^2}{8n}\right) - \frac{\pi^2}{6} \right) \end{aligned}$$



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$$Li_2(-2n) = \frac{\pi^2}{12} - \int_{\frac{1}{2n}}^1 \frac{\log(1+t) - \log t}{t} dt = -\frac{\pi^2}{6} - Li_2\left(-\frac{1}{2n}\right) - \frac{1}{2} \log^2(2n)$$

$$\Omega(a, n) = \frac{1}{2n-1} \left(\frac{1}{2} \log(2n) \log\left(\frac{(2n+1)^2}{8n}\right) + \log \log\left(\frac{(2n+1)^2}{8n}\right) \right)$$

$$\Omega(a) = \int_1^{2n} \frac{\log(ax)}{x^2 + (2n+1)x + 2n} dx = \frac{1}{4n-2} \cdot \log\left(\frac{(2n+1)^2}{8n}\right) \log(2na^2)$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Note that:

$$\int_p^q \frac{\log x}{(x+p)(x+q)} dx = \frac{\log(pq) \log\left(\frac{(p+q)^2}{4pq}\right)}{2(q-p)}, \quad (1)$$

Now,

$$\begin{aligned} \Omega(a) &= \int_1^{2n} \frac{\log(ax)}{x^2 + (2n+1)x + 2n} dx = \int_1^{2n} \frac{\log(ax)}{(x+2n)(x+1)} dx \stackrel{y=ax}{=} \\ &\stackrel{y=ax}{=} \int_a^{2na} \frac{\log x}{(x+2na)(x+a)} dx \stackrel{(1)}{=} \frac{\log(2na^2) \log\left(\frac{(2n+1)^2}{8n}\right)}{2(2n-1)} \end{aligned}$$

Solution 3 by Florică Anastase-Romania

$$\begin{aligned} \Omega(a) &= \int_1^{2n} \frac{\log(ax)}{x^2 + (2n+1)x + 2n} dx \stackrel{x=\frac{2n}{x}}{=} \int_{2n}^1 \frac{\log\left(\frac{2na}{x}\right)}{\left(\frac{2n}{x}\right)^2 + \frac{(2n+1)2n}{x} + 2n} \cdot \left(-\frac{2ndx}{x^2}\right) = \\ &= \int_1^{2n} \frac{\log(2na) - \log x}{x^2 + (2n+1)x + 2n} dx \\ 2\Omega(a) &= \int_1^{2n} \frac{\log(ax) + \log\left(\frac{2na}{x}\right)}{(x+1)(x+2n)} dx = \log(2na^2) \int_1^{2n} \frac{1}{(x+1)(x+2n)} dx = \\ &= \frac{\log(2na^2)}{2n-1} \cdot \int_1^{2n} \frac{(x+2n) - (x+1)}{(x+1)(x+2n)} dx = \frac{\log(2na^2)}{2n-1} \cdot \int_1^{2n} \left(\frac{1}{x+1} - \frac{1}{x+2n}\right) dx \end{aligned}$$



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$$\Omega(a) = \frac{\log(2na^2)}{4n-2} \cdot \log\left(\frac{t+1}{t+2n}\right)\Big|_1^{2n}$$

$$\Omega(a) = \frac{\log(2na^2)}{4n-2} \cdot \log\left(\frac{(2n+1)^2}{8n}\right)$$

1146. Prove that:

$$\int_0^\infty \frac{\log(1+x^4+x^6+x^{10})}{1+3x+3x^2+x^3} dx = \pi + \frac{\pi\sqrt{2}}{2} - \frac{\pi\sqrt{3}}{3} - \frac{5}{2}$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} \Omega &= \int_0^\infty \frac{\log(1+x^4+x^6+x^{10})}{1+3x+3x^2+x^3} dx = \int_0^\infty \frac{\log[1+x^4+x^6(1+x^4)]}{(1+x)^3} dx = \\ &= \int_0^\infty \frac{\log[(1+x^4)(1+x^6)]}{(1+x)^3} dx = \int_0^\infty \frac{\log(1+x^4) + \log(1+x^6)}{(1+x)^3} dx \stackrel{IBP}{=} \\ &= 2 \int_0^\infty \frac{x^3 dx}{(1+x)^2(1+x^4)} + 3 \int_0^\infty \frac{x^5 dx}{(1+x)^2(1+x^6)} \stackrel{t=\frac{1}{x}}{=} \\ &= 2 \int_0^\infty \frac{tdt}{(1+t)^2(1+t^4)} + 3 \int_0^\infty \frac{tdt}{(1+t)^2(1+t^6)} \\ \frac{t}{(1+t)^2(1+t^4)} &= \frac{A_1}{(1+t)^2} + \frac{A_2}{1+t} + \frac{A_3 t^3 + A_4 t^2 + A_5 t + A_6}{1+t^4} = \\ &= -\frac{1}{2(1+t)^2} - \frac{1}{2(1+t)} + \frac{t^3 - t + 2}{2(1+t^4)} \\ \frac{t}{(1+t)^2(1+t^6)} &= \frac{B_1}{(1+t)^2} + \frac{B_2}{1+t} + \frac{B_3 t^5 + B_4 t^4 + B_5 t^3 + B_6 t^2 + B_7 t + B_8}{1+t^6} = \\ &= -\frac{1}{2(1+t)^2} - \frac{1}{2(1+t)} + \frac{6t^5 - 3t^4 + 3t^2 - 6t + 9}{6(1+t^6)} \\ \Omega &= \left(-1 + \int_0^\infty \frac{2-t}{1+t^4} dt \right) - \frac{3}{2} + \frac{3}{2} \int_0^\infty \frac{3-2t+t^2-t^4}{1+t^6} dt = \end{aligned}$$

$$\begin{aligned}
 &= -\frac{5}{2} + \frac{1}{2} \int_0^\infty \frac{z^{-\frac{3}{4}}}{1+z} dz - \frac{1}{2} \int_0^\infty \frac{dz}{1+z^2} + \frac{3}{4} \int_0^\infty \frac{z^{-\frac{5}{6}}}{1+z} dz - \frac{1}{2} \int_0^\infty \frac{z^{-\frac{2}{3}}}{1+z} dz + \frac{1}{2} \int_0^\infty \frac{dz}{1+z^2} - \frac{1}{4} \int_0^\infty \frac{z^{-\frac{1}{6}}}{1+z} dz \\
 &= -\frac{5}{2} + \frac{1}{2} \cdot \frac{\pi}{\sin \frac{3\pi}{4}} + \frac{3}{4} \cdot \frac{\pi}{\sin \frac{5\pi}{6}} - \frac{1}{2} \cdot \frac{\pi}{\sin \frac{2\pi}{3}} - \frac{1}{4} \cdot \frac{\pi}{\sin \frac{\pi}{6}} = \\
 &= \pi + \frac{\pi\sqrt{2}}{2} - \frac{\pi\sqrt{3}}{3} - \frac{5}{2}
 \end{aligned}$$

1147.

$$\varnothing_n = \underbrace{\sqrt{2 + \sqrt{2 + \dots + \sqrt{2}}}}_{n-times}, \quad a_m = \frac{2m+1}{2(m^2+m)}$$

$$f_{m,n} = \int_{\frac{1}{2}\varnothing_n}^1 \left(\frac{1}{m+1-x^2} - \frac{1}{m+x^2} \right) \left(\frac{1}{m+1-x^2} + \frac{1}{m+x^2} - a_m \right) \frac{dx}{\sqrt{1-x^2}}$$

$$\text{Prove that: } f_{m,n} = \frac{\sqrt{2-\varnothing_{n-2}}}{(4m+2)^2 - 2 - \varnothing_{n-2}} \cdot 4a_m$$

Proposed by Mohamed Bouras-Fes-Morocco

Solution by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned}
 \varnothing_n^2 &= 2 + \varnothing_{n-1} \Rightarrow \varnothing_n^2 - 2 = \varnothing_{n-1}; (1) \\
 I_m(x) &= \int \left(\frac{1}{m+1-x^2} - \frac{1}{m+x^2} \right) \left(\frac{1}{m+1-x^2} + \frac{1}{m+x^2} - a_m \right) \frac{dx}{\sqrt{1-x^2}} = \\
 &= \int \left(\frac{1}{m+\cos^2 t} - \frac{1}{m+\sin^2 t} \right) \left(\frac{1}{m+\cos^2 t} + \frac{1}{m+\sin^2 t} - a_m \right) dt = \\
 &= - \int \frac{2m+1}{m^2+m+\frac{1}{4}\sin^2 2t} \cdot \frac{\cos 2t}{m^2+m+\frac{1}{4}\sin^2 2t} dt - a_m \int \left(\frac{1}{m+\cos^2 t} - \frac{1}{m+\sin^2 t} \right) dt \\
 &\stackrel{u=\frac{1}{2}\sin 2t}{=} - (2m+1) \int \frac{du}{(m^2+m+u^2)^2} + a_m \int \frac{du}{m^2+m+u^2} \\
 A(u) &= \int \frac{du}{m^2+m+u^2} = \frac{u}{m^2+m+u^2} + 2 \int \frac{u^2 du}{(m^2+m+u^2)^2}
 \end{aligned}$$



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$$\begin{aligned}
 A(u) &= \frac{u}{m^2 + m + u^2} - 2(m^2 + m) \int \frac{u^2 du}{(m^2 + m + u^2)^2} + 2A(u) \Rightarrow \\
 \int \frac{du}{(m^2 + m + u^2)^2} &= \frac{1}{2(m^2 + m)} \left(\frac{u}{m^2 + m + u^2} + \int \frac{du}{m^2 + m + u^2} \right) \\
 I_m(x) &= -\frac{2m+1}{2(m^2+m)} \left(\frac{u}{m^2+m+u^2} + \int \frac{du}{m^2+m+u^2} \right) + a_m \int \frac{du}{m^2+m+u^2} = \\
 &= -\frac{u}{m^2+m+u^2} \cdot a_m \stackrel{u=sintcost=x\sqrt{1-x^2}}{=} -\frac{x\sqrt{1-x^2}}{m^2+m+x^2(1-x^2)} \cdot a_m \\
 f_{m,n} &= \int_{\frac{1-\phi_n}{2}}^{\frac{1}{2}} \left(\frac{1}{m+1-x^2} - \frac{1}{m+x^2} \right) \left(\frac{1}{m+1-x^2} + \frac{1}{m+x^2} - a_m \right) \frac{dx}{\sqrt{1-x^2}} \\
 &= \frac{\frac{\phi_n}{2} \sqrt{1 - \frac{\phi_n^2}{4}}}{m^2 + m + \left(\frac{\phi_n}{2} \sqrt{1 - \frac{\phi_n^2}{4}} \right)^2} \cdot a_m = \frac{\phi_n \sqrt{4 - \phi_n^2}}{16(m^2 + m) + (\phi_n \sqrt{4 - \phi_n^2})^2} \cdot 4a_m \\
 \text{Or } \phi_n \sqrt{4 - \phi_n^2} &= \phi_n \sqrt{2 + 2 - \phi_n^2} = \sqrt{2 + \phi_{n-1}} \cdot \sqrt{2 - \phi_{n-1}} = \sqrt{2 + 2 - \phi_{n-1}^2} = \\
 &= \sqrt{2 - \phi_{n-2}} \text{ using (1)} \\
 f_{m,n} &= \frac{\sqrt{2 - \phi_{n-2}}}{16(m^2 + m) + 2 - \phi_{n-2}} \cdot 4a_m = \frac{\sqrt{2 - \phi_{n-2}}}{16m^2 + 2 \cdot 2 \cdot 4m + 4 - (2 + \phi_{n-2})} \cdot 4a_m = \\
 &= \frac{\sqrt{2 - \phi_{n-2}}}{(4m + 2)^2 - 2 - \phi_{n-2}} \cdot 4a_m \\
 f_{m,n} &= \frac{\sqrt{2 - \phi_{n-2}}}{(4m + 2)^2 - 2 - \phi_{n-2}} \cdot 4a_m
 \end{aligned}$$

1148. Prove that:

$$\int_0^\pi \sin^{-1} \left(\sec \left(\tan^{-1}(\sin x) \right) \right) \sin x dx = \pi + i \left(\frac{2\pi\sqrt{2\pi}}{\Gamma^2\left(\frac{1}{4}\right)} - \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} \right)$$

Proposed by Srinivasa Raghava-AIRMC-India



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Solution by Tobi Joshua-Nigeria

$$\begin{aligned}
 I &= \int_0^\pi \sin^{-1} \left(\sec \left(\tan^{-1}(\sin x) \right) \right) \sin x \, dx \\
 I &= \int_0^\pi \sin^{-1} \left(\sqrt{1 + \sin^2 x} \right) \sin x \, dx \stackrel{IBP}{=} \\
 &= -\cos x \cdot \sin^{-1} \left(\sqrt{1 + \sin^2 x} \right) \Big|_0^\pi + \frac{1}{i} \int_0^\pi \frac{\cos^2 x}{\sqrt{1 + \sin^2 x}} \, dx = \\
 &= \pi + \frac{1}{i} \int_0^\pi \frac{\cos^2 x}{\sqrt{1 + \sin^2 x}} \, dx = \pi - i \left[\int_0^\pi \frac{2}{\sqrt{1 + \sin^2 x}} \, dx - \int_0^\pi \sqrt{1 + \sin^2 x} \, dx \right]
 \end{aligned}$$

Using Incomplete Elliptical Integral of second kind:

$$E \left(\frac{z}{m} \right) = \int_0^z \sqrt{1 - m \sin^2 t} \, dz$$

Using Incomplete Elliptical Integral of first kind:

$$\begin{aligned}
 F \left(\frac{m}{z} \right) &= \int_0^z \frac{dz}{\sqrt{1 - m \sin^2 t}} \\
 I &= \pi - i \left[2F \left(\frac{\pi}{-1} \right) - E \left(\frac{\pi}{-1} \right) \right] = \pi - i[4K(-1) - 2E(-1)] = \\
 &= \pi - i \left[4 \left(\frac{\Gamma^2 \left(\frac{1}{4} \right)}{4\sqrt{2\pi}} \right) - 2 \left(\frac{\pi\sqrt{2\pi}}{\Gamma^2 \left(\frac{1}{4} \right)} + \frac{\Gamma^2 \left(\frac{1}{4} \right)}{4\sqrt{2\pi}} \right) \right] = \\
 &= \pi - i \left[\left(\frac{\Gamma^2 \left(\frac{1}{4} \right)}{\sqrt{2\pi}} \right) - \left(\frac{2\pi\sqrt{2\pi}}{\Gamma^2 \left(\frac{1}{4} \right)} + \frac{\Gamma^2 \left(\frac{1}{4} \right)}{2\sqrt{2\pi}} \right) \right]
 \end{aligned}$$

$$\int_0^\pi \sin^{-1} \left(\sec \left(\tan^{-1}(\sin x) \right) \right) \sin x \, dx = \pi + i \left(\frac{2\pi\sqrt{2\pi}}{\Gamma^2 \left(\frac{1}{4} \right)} - \frac{\Gamma^2 \left(\frac{1}{4} \right)}{2\sqrt{2\pi}} \right)$$



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1149. Find without softs:

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\sin x \left(\sqrt{\frac{1}{\cos x} + \tan x} + \sqrt{\frac{1}{\cos x} - \tan x} \right)}{\sqrt{\cos^5 x + \cos^6 x}} dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Abdul Hannan-Tezpur-India

Note that for $a, b > 0$ we have, $\sqrt{a} + \sqrt{b} = \sqrt{(\sqrt{a} + \sqrt{b})^2} = \sqrt{a + b + 2\sqrt{ab}}$

Now, in the interval $(0, \frac{\pi}{4})$ we have:

$$\begin{aligned} \frac{1}{\cos x} + \tan x &= \frac{1 + \sin x}{\cos x} > 0 \\ \frac{1}{\cos x} - \tan x &= \frac{1 - \sin x}{\cos x} > 0 \\ \sqrt{\frac{1}{\cos x} + \tan x} + \sqrt{\frac{1}{\cos x} - \tan x} &= \\ &= \sqrt{\frac{1}{\cos x} + \tan x + \frac{1}{\cos x} - \tan x + 2\sqrt{\left(\frac{1}{\cos x} + \tan x\right)\left(\frac{1}{\cos x} - \tan x\right)}} = \\ &= \sqrt{\frac{2}{\cos x} + 2\sqrt{\sec^2 x - \tan^2 x}} = \sqrt{\frac{2}{\cos x} + 2} = \sqrt{\frac{2(1 + \cos x)}{\cos x}} \end{aligned}$$

Therefore,

$$\begin{aligned} \Omega &= \int_0^{\frac{\pi}{4}} \frac{\sin x \left(\sqrt{\frac{1}{\cos x} + \tan x} + \sqrt{\frac{1}{\cos x} - \tan x} \right)}{\sqrt{\cos^5 x + \cos^6 x}} dx \\ &= \int_0^{\frac{\pi}{4}} \frac{\sin x \cdot \sqrt{\frac{2(1 + \cos x)}{\cos x}}}{\sqrt{(1 + \cos x)\cos^5 x}} dx \stackrel{u=\cos x}{=} \sqrt{2} \int_{\frac{1}{\sqrt{2}}}^1 \frac{du}{u^3} = \frac{1}{\sqrt{2}} \end{aligned}$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\begin{aligned}
 \Omega &= \int_0^{\frac{\pi}{4}} \frac{\sin x \left(\sqrt{\frac{1}{\cos x} + \tan x} + \sqrt{\frac{1}{\cos x} - \tan x} \right)}{\sqrt{\cos^5 x + \cos^6 x}} dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{\frac{\pi}{4} \sqrt{1 + \sin x} + \sqrt{1 - \sin x} \sin x}{\sqrt{\cos x}} dx = \\
 &= \int_0^{\frac{\pi}{4}} \frac{\frac{\pi}{4} \left(\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2} + \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)^2} \right) \sin x}{\sqrt{(1 + \cos x) \cos^6 x}} dx = \\
 &= \int_0^{\frac{\pi}{4}} \frac{2 \cos \frac{x}{2} \sin x}{\sqrt{2 \cos^2 \frac{x}{2} \cos^6 x}} dx = \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x} dx = -\sqrt{2} \int_0^{\frac{\pi}{4}} \frac{d(\cos x)}{\cos^3 x} = \frac{1}{\sqrt{2}}
 \end{aligned}$$

1150. Find without softs:

$$\Omega = \int_0^1 \frac{2x + 3\sqrt{1-x^2}}{5(1-x^2) + 4x\sqrt{1-x^2}} dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Yen Tung Chung-Taichung-Taiwan

$$\begin{aligned}
 \Omega &= \int_0^1 \frac{2x + 3\sqrt{1-x^2}}{5(1-x^2) + 4x\sqrt{1-x^2}} dx \stackrel{x=\sin\theta}{=} \int_0^{\frac{\pi}{2}} \frac{2\sin\theta + 3\cos\theta}{5\cos^2\theta + 4\sin\theta\cos\theta} \cdot \cos\theta d\theta = \\
 &= \int_0^{\frac{\pi}{2}} \frac{2\sin\theta + 3\cos\theta}{5\cos\theta + 4\sin\theta} d\theta = \frac{23}{41} \int_0^{\frac{\pi}{2}} \frac{5\cos\theta + 4\sin\theta}{5\cos\theta + 4\sin\theta} d\theta + \frac{2}{41} \int_0^{\frac{\pi}{2}} \frac{4\cos\theta - 5\sin\theta}{5\cos\theta + 4\sin\theta} d\theta = \\
 &= \frac{23}{41} \cdot \frac{\pi}{2} + \frac{2}{41} \log(5\cos\theta + 4\sin\theta) \Big|_0^{\frac{\pi}{2}} = \frac{23\pi}{82} + \frac{2}{41} \log\left(\frac{4}{5}\right)
 \end{aligned}$$

Solution 2 by Hussain Reza Zadah-Afghanistan



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$$\begin{aligned}
 \Omega &= \int_0^1 \frac{2x + 3\sqrt{1-x^2}}{5(1-x^2) + 4x\sqrt{1-x^2}} dx \stackrel{x=\sin u}{=} \int_0^{\frac{\pi}{2}} \frac{2\sin u + 3\cos u}{5\cos^2 u + 4\sin u \cos u} du = \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sin 2u + 3\cos^2 u}{5\cos^2 u + 2\sin 2u} du = \\
 &= \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} \frac{2\sin 2u + 5\cos^2 u}{5\cos^2 u + 2\sin 2u} du + \int_0^{\frac{\pi}{2}} \frac{\cos^2 u}{5\cos^2 u + 2\sin 2u} du \right] = \\
 &= \frac{\pi}{4} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos^2 u}{5\cos^2 u + 2\sin 2u} du = \\
 &= \frac{\pi}{4} + \frac{1}{2} \left[\frac{5}{41} u + \frac{4}{41} \log(5\cos^2 u + 2\sin 2u) \right]_0^{\frac{\pi}{2}} = \frac{23\pi}{82} + \frac{2}{41} \log\left(\frac{4}{5}\right)
 \end{aligned}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\begin{aligned}
 \Omega &= \int_0^1 \frac{2x + 3\sqrt{1-x^2}}{5(1-x^2) + 4x\sqrt{1-x^2}} dx \stackrel{x=\sin\theta}{=} \int_0^{\frac{\pi}{2}} \frac{2\sin\theta + 3\cos\theta}{5\cos^2\theta + 4\sin\theta\cos\theta} \cdot \cos\theta d\theta \\
 2\sin\theta + 3\cos\theta &= \alpha(5\cos\theta + 4\sin\theta) + \beta(-5\sin\theta + 4\cos\theta)
 \end{aligned}$$

Hence,

$$\begin{cases} 4\alpha - 5\beta = 2 \\ 5\alpha + 4\beta = 3 \end{cases} \Rightarrow \alpha = \frac{23}{41}, \beta = \frac{2}{41}$$

Thus,

$$\Omega = \frac{23}{41} \theta \Big|_0^{\frac{\pi}{2}} + \frac{2}{41} \log(5\cos\theta + 4\sin\theta) \Big|_0^{\frac{\pi}{2}} = \frac{23\pi}{82} + \frac{2}{41} \log\left(\frac{4}{5}\right)$$

Solution 4 by Orlando Irahola Ortega-Bolivia

$$\Omega = \int_0^1 \frac{2x + 3\sqrt{1-x^2}}{5(1-x^2) + 4x\sqrt{1-x^2}} dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{dx}{4x + 5\sqrt{1-x^2}}$$



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$$\begin{aligned}
 I_1 &= \int_0^1 \frac{dx}{\sqrt{1-x^2}} \stackrel{x^2=t \Rightarrow x=\sqrt{t}}{=} \frac{1}{2} \int_0^1 t^{\frac{1}{2}-1} (1-t)^{\frac{1}{2}-1} dt = \frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{\pi}{2} \\
 I_2 &= \int_0^1 \frac{dx}{4x + 5\sqrt{1-x^2}} \stackrel{x=\frac{1}{t}}{=} \int_1^\infty \frac{dt}{t(5\sqrt{t^{2-1}+4})} = \\
 &= \int_1^\infty \frac{tdt}{(t^2-1)(5\sqrt{t^2-1}+4)} \stackrel{z=\sqrt{t^2-1}}{=} \int_0^\infty \frac{zdz}{(5z+4)(z^2+1)} \\
 41I_2 &= \int_0^\infty \frac{41zdz}{(5z+4)(z^2+1)} = \int_0^\infty \frac{41z + 20z^2 + 20 - 20(z^2+1)}{(5z+4)(z^2+1)} dz = \\
 &= \int_0^\infty \frac{(20z^2 + 25z) + (16z + 20) - 20(z^2+1)}{(5z+4)(z^2+1)} dz = \\
 &= \int_0^\infty \frac{5z(4z+5) + 4(4z+5) - 20(z^2+1)}{(5z+4)(z^2+1)} dz = \\
 &= \int_0^\infty \frac{(4z+5)(5z+4) - 20(z^2+1)}{(5z+4)(z^2+1)} dz = \\
 &= \int_0^\infty \frac{4z+5}{z^2+1} dz - 20 \int_0^\infty \frac{dz}{5z+4} = \left[2 \int \frac{2zdz}{z^2+1} - 20 \int \frac{dz}{5z+4} + 5 \int \frac{dz}{z^2+1} \right]_0^\infty = \\
 &= [2\log(z^2+1) + 4\log(5z+4) + 5\tan^{-1}z]_0^\infty = \\
 &= \left[\log\left(\frac{z^2+1}{25z^2+40z+16}\right)^2 + 5\tan^{-1}z \right]_0^\infty = 4\log\left(\frac{4}{5}\right) + \frac{5\pi}{2} \\
 I_2 &= \frac{4}{41} \log\left(\frac{4}{5}\right) + \frac{5\pi}{82}. \text{ Therefore,} \\
 I &= \frac{1}{2}(I_1 + I_2) = \frac{23\pi}{82} + \frac{2}{41} \log\left(\frac{4}{5}\right)
 \end{aligned}$$

Solution 5 by Abner Chinga Bazo-Peru

$$\Omega = \int_0^1 \frac{2x + 3\sqrt{1-x^2}}{5(1-x^2) + 4x\sqrt{1-x^2}} dx$$



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$$\text{Let } x = \frac{2y}{1+y^2}, dx = \frac{2(1-y^2)}{(1+y^2)^2} dy$$

$$\begin{aligned}
I &= \int \frac{2\left(\frac{2y}{1+y^2}\right) + 3\sqrt{1 - \left(\frac{2y}{1+y^2}\right)^2}}{5\left(1 - \left(\frac{2y}{1+y^2}\right)^2\right) + 4\left(\frac{2y}{1+y^2}\right)\sqrt{1 - \left(\frac{2y}{1+y^2}\right)^2}} \left(\frac{2(1-y^2)}{(1+y^2)^2}\right) dy = \\
&= \int \frac{\frac{4y}{1+y^2} + \frac{3(1-y^2)}{1+y^2}}{5\left(\frac{2y}{1+y^2}\right)^2 + 4\left(\frac{2y}{1+y^2}\right)\left(\frac{1-y^2}{1+y^2}\right)} \left(\frac{2(1-y^2)}{(1+y^2)^2}\right) dy = \\
&= \int \frac{2(3y^2 - 4y - 3)}{(5y^2 - 8y - 5)(1+y^2)} dy = 2 \int \left(\frac{23-2y}{41(1+y^2)} + \frac{10y-8}{41(5y^2-8y-5)} \right) dy = \\
&= \frac{2}{41} \int \left(\frac{23}{1+y^2} - \frac{2y}{1+y^2} + \frac{10y-8}{5y^2-8y-5} \right) dy = \\
&= \frac{2}{41} (23 \tan^{-1} y - \log(y^2 + 1) + \log|5y^2 - 8y - 5|) + C \\
\Omega &= \int_0^1 \frac{2x + 3\sqrt{1-x^2}}{5(1-x^2) + 4x\sqrt{1-x^2}} dx \\
&= \frac{2}{41} \left[23 \tan^{-1} y - \log(y^2 + 1) + \log|5y^2 - 8y - 5| \right]_0^1 \\
&= \frac{23\pi}{82} + \frac{2}{41} \log\left(\frac{4}{5}\right)
\end{aligned}$$

1151.

If $f: [0, \infty) \rightarrow \mathbb{R}$, $f(x) = \int_0^x \frac{\log(1+t)}{1+t^2} dt$ then find:

$$\Omega = \int_0^1 xf(x) dx$$

Proposed by Alex Szoros-Romania

Solution 1 by Florică Anastase-Romania



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$$\begin{aligned}
 & \int_0^1 \frac{\log(1+t)}{1+t^2} dt \stackrel{t=tanu}{=} \int_0^{\frac{\pi}{4}} \frac{\log(1+tanu)}{\frac{1}{\cos^2 u}} \cdot \frac{du}{\cos^2 u} = \\
 & = \int_0^{\frac{\pi}{4}} \log\left(\frac{\sin u + \cos u}{\cos u}\right) du = \int_0^{\frac{\pi}{4}} \log\left[\frac{\sqrt{2}\cos\left(\frac{\pi}{4}-u\right)}{\cos u}\right] du = \\
 & = \int_0^{\frac{\pi}{4}} \log\sqrt{2} du + \int_0^{\frac{\pi}{4}} \log\left[\cos\left(\frac{\pi}{4}-u\right)\right] du - \int_0^{\frac{\pi}{4}} \log(\cos u) du = \\
 & \int_0^{\frac{\pi}{4}} \log\left[\cos\left(\frac{\pi}{4}-u\right)\right] du \stackrel{\frac{\pi}{4}-u=v}{=} - \int_0^{\frac{\pi}{4}} \log(\cos v) dv = v \\
 & \int_0^1 \frac{\log(1+t)}{1+t^2} dt = \frac{\pi}{8} \log 2 \Rightarrow f(1) = \frac{\pi}{8} \log 2 \\
 \Omega &= \int_0^1 xf(x) dx \stackrel{IBP}{=} \frac{x^2}{2} f(x) \Big|_0^1 - \frac{1}{2} \int_0^1 x^2 \cdot \frac{\log(1+x)}{1+x^2} dx = \\
 &= \frac{1}{2} f(1) - \frac{1}{2} \int_0^1 (1+x^2-1) \cdot \frac{\log(1+x)}{1+x^2} dx = \\
 &= \frac{\pi}{16} \log 2 - \frac{x}{2} \log(1+x) \Big|_0^1 + \frac{1}{2} \int_0^1 \frac{\log(1+x)}{1+x^2} dx = \\
 &= \frac{\pi-8}{8} \log 2 + \frac{1}{2}
 \end{aligned}$$

Solution 2 by Elli Asaad-Afghanistan

$$\begin{aligned}
 A &= \int_0^1 \frac{\log(1+t)}{1+t^2} dt \stackrel{t=tan\theta}{=} \int_0^{\frac{\pi}{4}} \log(1+tan\theta) d\theta = \\
 &= \int_0^{\frac{\pi}{4}} \log(\sin\theta + \cos\theta) d\theta - \int_0^{\frac{\pi}{4}} \log(\cos\theta) d\theta =
 \end{aligned}$$



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$$= \int_0^{\frac{\pi}{4}} \log \sqrt{2} d\theta + \int_0^{\frac{\pi}{4}} \log \left(\cos \left(\theta - \frac{\pi}{4} \right) \right) d\theta - \int_0^{\frac{\pi}{4}} \log (\cos \theta) d\theta = \frac{\pi}{8} \log 2$$

$$\begin{aligned} \Omega &= \int_0^1 x f(x) dx \stackrel{IBP}{=} \frac{x^2}{2} f(x) \Big|_0^1 - \frac{1}{2} \int_0^1 x^2 \cdot \frac{\log(1+x)}{1+x^2} dx = \\ &= \frac{1}{2} f(1) - \frac{1}{2} \int_0^1 (1+x^2-1) \cdot \frac{\log(1+x)}{1+x^2} dx = \\ &= f(1) - \frac{1}{2} \int_0^1 \log(1+x) dx = A - B; \end{aligned}$$

$$B = \frac{1}{2} \int_0^1 \log(1+t) dt = \frac{1}{2} [(1+t)\log(1+t) - t]_0^1 = \frac{1}{2} (2\log 2 - 1)$$

$$\Omega = A - B = \frac{\pi - 8}{8} \log 2 + \frac{1}{2}$$

Solution 3 by Praveen Kumar Kotra-India

$$\begin{aligned} f'(x) &= \frac{\log(1+x)}{1+x^2} \\ f(1) &= \int_0^1 \frac{\log(1+x)}{1+x^2} dx \stackrel{t=tan\theta}{=} \int_0^{\frac{\pi}{4}} \log(1+\tan\theta) d\theta = \\ &= \int_0^{\frac{\pi}{4}} \log(\sin\theta + \cos\theta) d\theta - \int_0^{\frac{\pi}{4}} \log(\cos\theta) d\theta = \\ &= \int_0^{\frac{\pi}{4}} \log \sqrt{2} d\theta + \int_0^{\frac{\pi}{4}} \log \left(\cos \left(\theta - \frac{\pi}{4} \right) \right) d\theta - \int_0^{\frac{\pi}{4}} \log (\cos \theta) d\theta = \frac{\pi}{8} \log 2 \\ \int_0^1 x f(x) dx &\stackrel{IBP}{=} \frac{x^2}{2} f(x) \Big|_0^1 - \frac{1}{2} \int_0^1 x^2 \cdot \frac{\log(1+x)}{1+x^2} dx = \end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{2}f(1) - \frac{1}{2} \int_0^1 (1 + x^2 - 1) \cdot \frac{\log(1+x)}{1+x^2} dx = \\
 &= \frac{\pi}{16} \log 2 - \frac{x}{2} \log(1+x) \Big|_0^1 + \frac{1}{2} \int_0^1 \frac{\log(1+x)}{1+x^2} dx = \\
 &= \frac{\pi - 8}{8} \log 2 + \frac{1}{2}
 \end{aligned}$$

Solution 4 by Precious Itsuokor-Nigeria

$$\begin{aligned}
 \Omega &= \int_0^1 xf(x)dx = \int_t^1 x \int_0^x \frac{\log(1+t)}{1+t^2} dx dt \stackrel{\text{Fubini}}{=} \int_0^1 \frac{\log(1+t)}{1+t^2} \int_t^1 x dx dt = \\
 &= \int_0^1 \frac{\log(1+t)}{1+t^2} \cdot \left(\frac{x^2}{2}\right)_t^1 dt = \frac{1}{2} \int_0^1 \frac{\log(1+t)}{1+t^2} (1-t^2) dx = \\
 &= \frac{1}{2} \int_0^1 \frac{\log(1+t)}{1+t^2} dt - \frac{1}{2} \int_0^1 \frac{t^2 \log(1+t)}{1+t^2} dt = \\
 &= \frac{1}{2} \int_0^1 \frac{\log(1+t)}{1+t^2} dt - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+t^2}\right) \log(1+t) dt = \\
 &= \frac{1}{2} \int_0^1 \frac{\log(1+t)}{1+t^2} dt - \frac{1}{2} \int_0^1 \log(1+t) dt + \frac{1}{2} \int_0^1 \frac{\log(1+t)}{1+t^2} dt = \\
 &= \int_0^1 \frac{\log(1+t)}{1+t^2} dt - \frac{1}{2} ((1+t) \log(1+t) - (1+t)) \Big|_0^1 = \\
 &= \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) d\theta - \log 2 + \frac{1}{2} = \\
 &= \int_0^{\frac{\pi}{4}} \log(\sin \theta + \cos \theta) d\theta - \int_0^{\frac{\pi}{4}} \log(\cos \theta) d\theta - \log 2 + \frac{1}{2} =
 \end{aligned}$$



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$$\begin{aligned}
 &= \int_0^{\frac{\pi}{4}} \log \sqrt{2} d\theta + \int_0^{\frac{\pi}{4}} \log \left(\cos \left(\theta - \frac{\pi}{4} \right) \right) d\theta - \int_0^{\frac{\pi}{4}} \log (\cos \theta) d\theta - \log 2 + \frac{1}{2} = \\
 &= \frac{\pi}{8} \log 2 - \log 2 + \frac{1}{2}
 \end{aligned}$$

1152. Find without softs:

$$\Omega = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin^2(7x) + \cos^2(10x)}{\sin^2 x} dx$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

$$\begin{aligned}
 \text{Consider } & \frac{-\sin^2(7x) + \sin^2(10x)}{\sin^2 x} = \frac{\sin(17x)\sin(3x)}{\sin^2 x} = \frac{\sin(17x)}{\sin x} \cdot \frac{\sin(3x)}{\sin x} \\
 \text{But } & \frac{\sin(17x)}{\sin x} = \frac{\sin(17x) - \sin(15x) + \sin(15x) - \sin(13x) + \dots + \sin(3x) - \sin x + \sin x}{\sin x} = \\
 &= 2\cos(16x) + 2\cos(14x) + \dots + 2\cos(2x) + 1 \\
 &\quad \frac{\sin(3x)}{\sin x} = 2\cos(2x) + 1 \\
 \frac{\sin(17x)}{\sin x} \cdot \frac{\sin(3x)}{\sin x} &= 2\cos(18x) + 4\cos(16x) + 6 \sum_{k=1}^7 \cos(2kx) + 3
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \Omega &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin^2(7x) + \cos^2(10x)}{\sin^2 x} dx \\
 &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\csc^2 x - 3 - 2\cos(18x) - 4\cos(16x) - 6 \sum_{k=1}^7 \cos(2kx) \right] dx \\
 &= \left[-\cot x - 3x - \frac{1}{9} \sin(18x) - \frac{1}{4} \sin(16x) - 3 \sum_{k=1}^7 \frac{1}{k} \sin(2kx) \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}
 \end{aligned}$$



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$$\begin{aligned}
 &= -\frac{3\pi}{2} - \frac{1}{\sqrt{3}} + \pi + \frac{1}{9\sin(6\pi)} + \frac{1}{4}\sin\left(\frac{16\pi}{3}\right) + 3 \sum_{k=1}^7 \frac{1}{k}\sin\left(\frac{2k\pi}{3}\right) \\
 &= -\frac{\pi}{2} + \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{8} + 3 \left[\frac{1}{1} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + 0 + \frac{1}{4} \cdot \frac{\sqrt{3}}{2} - \frac{1}{5} \cdot \frac{\sqrt{3}}{2} + \frac{1}{7} \cdot \frac{\sqrt{3}}{2} \right] \\
 &= -\frac{\pi}{2} - \sqrt{3}\left(-\frac{1}{3} + \frac{1}{8}\right) + \frac{3\sqrt{3}}{2} \cdot \frac{97}{140} = \frac{131\sqrt{3}}{105} - \frac{\pi}{2}
 \end{aligned}$$

1153.

If $I_n(m) = \int_0^1 \frac{\log^n x}{1+x^m} dx$, $\forall m > 0, n \geq 0; m, n \in \mathbb{Z}$ then prove:

$$I_n(m) = \frac{1}{(2m)^{n+1}} \left[\psi^{(n)}\left(\frac{1+m}{2m}\right) - \psi^{(n)}\left(\frac{1}{2m}\right) \right]$$

Evaluate $I_2(2)$

Proposed by Akerele Olofin-Nigeria

Solution by Rana Ranino-Setif-Algerie

$$\begin{aligned}
 &\text{Set } x = e^{-\frac{t}{2m}} \\
 I_n(m) &= \frac{(-1)^n}{(2m)^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{t}{2m}}}{1 + e^{-\frac{t}{2}}} dt = \frac{(-1)^n}{(2m)^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{t}{2m}} \left(1 - e^{-\frac{t}{2}}\right)}{1 + e^{-\frac{t}{2}} \left(1 - e^{-\frac{t}{2}}\right)} dt \\
 I_n(m) &= \frac{(-1)^n}{(2m)^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{t}{2m}}}{1 - e^{-t}} dt - \frac{(-1)^n}{(2m)^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{1+m}{2m}t}}{1 + e^{-t}} dt \\
 &= \frac{(-1)^{n+1}}{(2m)^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{1+m}{2m}t}}{1 + e^{-t}} dt - \frac{(-1)^{n+1}}{(2m)^{n+1}} \int_0^\infty \frac{t^n e^{-\frac{t}{2m}}}{1 - e^{-t}} dt \\
 \therefore \psi^{(k)} &= (-1)^{k+1} \int_0^\infty \frac{t^k e^{-zt}}{1 - e^{-t}} dt \\
 I_n(m) &= \frac{1}{(2m)^{n+1}} \left[\psi^{(n)}\left(\frac{1+m}{2m}\right) - \psi^{(n)}\left(\frac{1}{2m}\right) \right]
 \end{aligned}$$



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$$\begin{aligned}
 I_2(2) &= \frac{1}{64} \left[\psi^{(2)}\left(\frac{3}{4}\right) - \psi^{(2)}\left(\frac{1}{4}\right) \right] = \frac{1}{64} \left[\psi^{(2)}\left(\frac{3}{4}\right) - \psi^{(2)}\left(1 - \frac{3}{4}\right) \right] = \\
 &= \frac{1}{64} \cdot \frac{d^2}{ds^2} [-\pi \cot(\pi s)]_{s=\frac{3}{4}} = \frac{\pi^2}{64} \cdot \frac{d}{ds} [\csc^2(\pi s)]_{s=\frac{3}{4}} = \\
 &= -\frac{\pi^3}{32} [\cot(\pi s) \csc^2(\pi s)]_{s=\frac{3}{4}} = \frac{\pi^3}{16} \\
 &\int_0^1 \frac{\log^2 x}{1+x^2} dx = \frac{\pi^3}{16}
 \end{aligned}$$

1154. Find a closed form:

$$\Omega = \left(\int_{\frac{\sqrt{3}}{3}}^1 \frac{x \log x dx}{3x^4} \right) \left(\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{x \log x dx}{x^4 + 1} \right) \left(\int_1^{\sqrt{3}} \frac{x \log x dx}{x^4 + 3} \right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Timson Azeez Folorunsho-Nigeria

$$\begin{aligned}
 A &= \int_{\frac{\sqrt{3}}{3}}^1 \frac{x \log x dx}{3x^4} \stackrel{y=x^2}{=} \frac{1}{4} \int_{\frac{1}{3}}^1 \frac{\log y dy}{3y^2 + 1} \stackrel{y=\frac{u}{\sqrt{3}}}{=} \frac{1}{4\sqrt{3}} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{(\log u - \log \sqrt{3}) du}{u^2 + 1} = \\
 &= \frac{1}{4\sqrt{3}} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{\log u}{u^2 + 1} du - \frac{1}{4\sqrt{3}} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{\log \sqrt{3}}{u^2 + 1} du = \\
 &= \frac{1}{4\sqrt{3}} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{\log u}{u^2 + 1} du - \frac{\pi \log \sqrt{3}}{12\sqrt{3}} + \frac{\pi \log \sqrt{3}}{24\sqrt{3}}; \quad (1)
 \end{aligned}$$

$$u \rightarrow \frac{1}{u} \Rightarrow A = -\frac{1}{4\sqrt{3}} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{\log u}{u^2 + 1} du - \frac{\pi \log \sqrt{3}}{12\sqrt{3}} + \frac{\pi \log \sqrt{3}}{24\sqrt{3}}; \quad (2)$$

$$2A = -\frac{\pi \log \sqrt{3}}{12\sqrt{3}} + \frac{\pi \log \sqrt{3}}{24\sqrt{3}} - \frac{\pi \log \sqrt{3}}{12\sqrt{3}} + \frac{\pi \log \sqrt{3}}{24\sqrt{3}}$$



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$$A = -\frac{\pi \log \sqrt{3}}{12\sqrt{3}} + \frac{\pi \log \sqrt{3}}{24\sqrt{3}} = -\frac{\pi \log \sqrt{3}}{24\sqrt{3}}$$

$$B = \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{x \log x dx}{x^4 + 1} \stackrel{y=x^2}{=} \frac{1}{2} \int_{\frac{1}{3}}^3 \frac{\log \sqrt{y}}{y^2 + 1} dy = \frac{1}{4} \int_{\frac{1}{3}}^3 \frac{\log y}{y^2 + 1} dy; \quad (1)$$

$$y \rightarrow \frac{1}{y} \Rightarrow B = -\frac{1}{4} \int_{\frac{1}{3}}^3 \frac{\log y}{y^2 + 1} dy \Rightarrow 2B = 0 \Rightarrow B = 0$$

$$C = \int_1^{\sqrt{3}} \frac{x \log x dx}{x^4 + 3} \stackrel{y=x^2}{=} \frac{1}{2} \int_1^{\sqrt{3}} \frac{\log \sqrt{y}}{y^2 + (\sqrt{3})^2} dy = \frac{1}{4} \int_1^3 \frac{\log y}{y^2 + (\sqrt{3})^2} dy \stackrel{y=u\sqrt{3}}{=}$$

$$= \frac{\sqrt{3}}{4} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\log(u\sqrt{3}) du}{3u^2 + 3} = \frac{\sqrt{3}}{12} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\log u}{u^2 + 1} du + \frac{\sqrt{3}}{12} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\log \sqrt{3}}{u^2 + 1} du =$$

$$= \frac{\sqrt{3}}{12} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\log u}{u^2 + 1} du + \frac{\sqrt{3}}{12} \log \sqrt{3} \cdot \tan^{-1} u \Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$C = \frac{\sqrt{3}}{12} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\log u}{u^2 + 1} du + \frac{\pi \sqrt{3} \log \sqrt{3}}{36} - \frac{\pi \sqrt{3} \log \sqrt{3}}{72} =$$

$$= \frac{\sqrt{3}}{12} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\log u}{u^2 + 1} du + \frac{\pi \sqrt{3} \log \sqrt{3}}{72}; \quad (1)$$

$$u \rightarrow \frac{1}{u} \Rightarrow C = -\frac{\sqrt{3}}{12} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\log u}{u^2 + 1} du + \frac{\pi \sqrt{3} \log \sqrt{3}}{72}; \quad (2)$$

$$\Omega = ABC = 0$$

Solution 2 by Ravi Prakash-New Delhi-India

Let:



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$$I_1 = \int_{\frac{\sqrt{3}}{3}}^1 \frac{x \log x dx}{3x^4}; I_2 = \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{x \log x dx}{x^4 + 1}; I_3 = \int_1^{\sqrt{3}} \frac{x \log x dx}{x^4 + 3}$$

We show that $I_1 = -I_3; I_2 = 0, I_3 = \frac{\pi \sqrt{3} \log 3}{144}$

$$\begin{aligned} I_2 &= \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{x \log x dx}{x^4 + 1} \stackrel{x=\frac{1}{t}}{=} \int_{\sqrt{3}}^{\frac{1}{3}} \frac{\frac{1}{t} \left(-\log t \right)}{\frac{1}{t^4} + 1} \left(-\frac{1}{t^2} \right) dt = \\ &= -\frac{\sqrt{3}}{3} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{t \log t}{t^4 + 1} dt = -I_2 \Rightarrow 2I_2 = 0 \Rightarrow I_2 = 0 \end{aligned}$$

$$I_1 = \int_{\frac{\sqrt{3}}{3}}^1 \frac{x \log x dx}{3x^4} \stackrel{x=\frac{1}{t}}{=} \int_{\sqrt{3}}^{\frac{1}{3}} \frac{\frac{1}{t} \left(-\log t \right)}{\frac{3}{t^4} + 1} \left(-\frac{1}{t^2} \right) dt = -\int_1^{\sqrt{3}} \frac{t \log t}{t^4 + 3} dt = -I_3$$

$$I_3 = \int_1^{\sqrt{3}} \frac{x \log x dx}{x^4 + 3} \stackrel{x^2=t\sqrt{3}}{=} \frac{\sqrt{3}}{2} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{\log(\sqrt[4]{3t^2})}{3t^2 + 3} dt = \frac{\sqrt{3}}{24} \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{\log 3}{t^2 + 1} dt + \frac{\sqrt{3}}{12} I_4$$

$$I_4 = \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{\log t}{t^2 + 1} dt \stackrel{y=\frac{1}{t}}{=} \int_{\sqrt{3}}^{\frac{1}{3}} \frac{\left(-\log t \right)}{1 + \frac{1}{t^2}} \left(-\frac{1}{t^2} \right) dt = -I_4 \Rightarrow I_4 = 0$$

$$I_3 = \frac{\sqrt{3}}{24} \log 3 \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi \sqrt{3}}{144} \log 3$$

Therefore, $I_1 I_2 I_3 = 0$

Solution 3 by Adrian Popa-Romania

$$I_1 = \int_{\frac{\sqrt{3}}{3}}^1 \frac{x \log x dx}{3x^4}; I_2 = \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{x \log x dx}{x^4 + 1}; I_3 = \int_1^{\sqrt{3}} \frac{x \log x dx}{x^4 + 3}$$



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$$\begin{aligned}
 I_2 &= \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{x \log x dx}{x^4 + 1} \stackrel{x=\frac{1}{t}}{=} \int_{\sqrt{3}}^{\frac{\sqrt{3}}{3}} \frac{\frac{1}{t^3} \frac{1}{t} (-\log t)}{\frac{1}{t^4} + 1} \left(-\frac{1}{t^2} \right) dt = \\
 &= -\frac{\sqrt{3}}{3} \int_{\sqrt{3}}^{\frac{\sqrt{3}}{3}} \frac{t \log t}{t^4 + 1} dt = -I_2 \Rightarrow 2I_2 = 0 \Rightarrow I_2 = 0
 \end{aligned}$$

$$\Omega = I_1 I_2 I_3 = 0$$

1155. Find without any software:

$$\Omega = \int (e^{-x} + \cot(e^x)) \cot(e^x) dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Șerban-Romania

$$\begin{aligned}
 \Omega &= \int (e^{-x} + \cot(e^x)) \cot(e^x) dx = \int (e^{-x} \cot(e^x) + \cot^2(e^x)) dx = \\
 &= \int (e^{-x} \cot(e^x) + \cot^2(e^x) + 1 - 1) dx = \\
 &= \int \left(e^{-x} \cot(e^x) + \frac{1}{\sin^2 e^x} \right) dx - \int dx = \\
 &= \int \left[-(e^{-x})' \cot(e^x) - (\cot(e^x))' e^{-x} \right] dx - x = \\
 &= -e^{-x} \cot(e^x) - x + C
 \end{aligned}$$

Solution 2 by Adrian Popa-Romania

$$\begin{aligned}
 \Omega &= \int (e^{-x} + \cot(e^x)) \cot(e^x) dx = \int e^{-x} \cot(e^x) dx + \int \cot^2(e^x) dx \\
 \cot^2(e^x) &= \frac{\cos^2(e^x)}{\sin^2(e^x)} = \frac{1 - \sin^2(e^x)}{\sin^2(e^x)} = \frac{1}{\sin^2(e^x)} - 1 = \frac{(-\cot(e^x))'}{e^x} - 1 \\
 &= -e^{-x} (\cot(e^x))' - 1
 \end{aligned}$$

$$\begin{aligned}
 \Omega &= \int e^{-x} \cot(e^x) dx - \int e^{-x} (\cot(e^x))' dx - \int dx = \\
 &= \int e^{-x} \cot(e^x) dx - \left(e^{-x} \cot(e^x) - \int (e^{-x})' \cot(e^x) dx \right) - x =
 \end{aligned}$$



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$$\begin{aligned}
 &= \int e^{-x} \cot(e^x) dx - e^{-x} \cot(e^x) - \int e^{-x} \cot(e^x) dx - x = \\
 &= -e^{-x} \cot(e^x) - x + C
 \end{aligned}$$

Solution 3 by Yen Tung Chung-Taichung-Taiwan

$$\begin{aligned}
 \Omega &= \int (e^{-x} + \cot(e^x)) \cot(e^x) dx = \int e^{-x} \cot(e^x) dx + \int \cot^2(e^x) dx \\
 &\quad \left(u = \cot(e^x); \quad dv = e^{-x} dx \right) \\
 &\quad \left(du = -e^{-x} \csc^2(e^x) dx; \quad v = -e^{-x} \right) \\
 &= -e^{-x} \cot(e^x) - \int \csc^2(e^x) dx - \int (\csc^2(e^x) - 1) dx = \\
 &= -e^{-x} \cot(e^x) - x + C
 \end{aligned}$$

1156. Find without any software:

$$\Omega(a) = \int_0^a \sec x \cdot \sec(a-x) dx, \quad a \in (0, \frac{\pi}{2})$$

Proposed by Muhhamad Menal-Dhaka-Bangladesh

Solution by Daniel Sitaru-Romania

$$\begin{aligned}
 \Omega(a) &= \int_0^a \sec x \cdot \sec(a-x) dx = \csc a \int_0^a \frac{\sin a}{\cos x \cdot \cos(a-x)} dx = \\
 &= \csc a \int_0^a \frac{\sin(x+a-x)}{\cos x \cdot \cos(a-x)} dx = \csc a \int_0^a \frac{\sin(a-x)\cos x + \sin x \cos(a-x)}{\cos x \cdot \cos(a-x)} dx = \\
 &= \csc a \left(\int_0^a \frac{\sin(a-x)}{\cos(a-x)} dx + \int_0^a \frac{\sin x}{\cos x} dx \right) = \csc a (-\ln(\cos a) - \ln(\cos a)) + C = \\
 &= \csc a \cdot \ln(\cos^{-2} a) + C = \csc a \cdot \ln(\sec^2 a) + C
 \end{aligned}$$

1157. Find a closed form:

$$\Omega = \int_{-\infty}^{\infty} \sin\left(\frac{1}{1+x^2}\right) dx$$

Proposed by Razi Talal Naji-Baghdad-Iraq



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Solution by proposer

$$\begin{aligned}
\Omega &= \int_{-\infty}^{\infty} \sin\left(\frac{1}{1+x^2}\right) dx = 2 \int_0^{\infty} \sin\left(\frac{1}{1+x^2}\right) dx = \\
&= 2 \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{1+x^2}\right)^{2n+1}}{(2n+1)!} dx = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\infty} \frac{1}{(1+x^2)^{2n+1}} dx \stackrel{x=tan\theta}{=} \\
&= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta d\theta}{(1+tan\theta)^{2n+1}} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\frac{\pi}{2}} \frac{\sec^2 \theta d\theta}{(\sec^2 \theta)^{2n+1}} = \\
&= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\frac{\pi}{2}} \sec^{-4n} \theta d\theta = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\frac{\pi}{2}} \cos^{4n} \theta d\theta = \\
&= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\frac{\pi}{2}} (\cos^2 \theta)^{\frac{4n}{2}} d\theta = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\frac{\pi}{2}} (1 - \sin^2 \theta)^{2n} d\theta \stackrel{u=\sin^2 \theta}{=} \\
&= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 (1-u)^{2n} \cdot \frac{1}{2} u^{-\frac{1}{2}} du = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 u^{-\frac{1}{2}} (1-u)^{2n} (1-u)^{-\frac{1}{2}} du = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 u^{\frac{1}{2}-1} (1-u)^{2n-\frac{1}{2}} du = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{2} F_1 \left[\frac{1}{2} - 2n, \frac{1}{2}, \frac{3}{2}; u \right] \Big|_0^1 = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} F_1 \left[\frac{1}{2} - 2n, \frac{1}{2}, \frac{3}{2}; 1 \right] \\
F_1[a; b; c; 1] &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c)} \\
F_1 \left[\frac{1}{2} - 2n, \frac{1}{2}, \frac{3}{2}; 1 \right] &= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}-\frac{1}{2}+2n-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{2}+2n\right)\Gamma(1)} = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma(2n+1)}
\end{aligned}$$



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$$\Omega = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(2n + \frac{1}{2}\right)}{\Gamma(2n+1)} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{\Gamma\left(2n + \frac{1}{2}\right)}{\Gamma(2n+1)} \cdot \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$(\therefore \Gamma(2n+1) = (2n)!!)$$

$$\Omega = \sqrt{\pi} \Gamma\left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! (2n)!} \cdot \frac{\Gamma\left(2n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n}}{(2n+1)! (2n)!}$$

$$(x_n)_{mn} = m^{mn} \prod_{k=0}^{m-1} \left(\frac{x+k}{m}\right)_n \Rightarrow \left(\frac{1}{2}\right)^{2n} = 2^{2n} \prod_{k=0}^{m-1} \left(\frac{\frac{1}{2}+k}{2}\right)_n = 2^{2n} \left(\frac{\frac{1}{2}+0}{2}\right)_n \left(\frac{\frac{1}{2}+1}{2}\right)_n$$

$$\Omega = \pi \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(2n+1)! (2n)!} =$$

$$= \pi \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(2n+1) 2^n \cdot n! (2n-1)!! 2^n \cdot n! (2n-1)!!} =$$

$$= \pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{2^{2n} \frac{(2n+1)!!}{2^n} \cdot n! \frac{(2n-1)!!}{2^n} \cdot n!} = \pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{\left(\frac{3}{2}\right)_n (1)_n \left(\frac{1}{2}\right)_n} \cdot \frac{\left(-\frac{1}{4}\right)_n}{n!} = \\ = \pi F_3 \left[\frac{1}{4}, \frac{3}{2}; \frac{3}{2}, 1, \frac{1}{2}; -\frac{1}{4} \right]$$

$$\Omega = \int_{-\infty}^{\infty} \sin\left(\frac{1}{1+x^2}\right) dx = \pi F_3 \left[\frac{1}{4}, \frac{3}{2}; \frac{3}{2}, 1, \frac{1}{2}; -\frac{1}{4} \right] \cong 2,95226276120165636$$

1158. If $\operatorname{Re}(n) > 0$

$$U(n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1+x^2}{1+y^2} e^{-ny^2-x^2} dy dx$$

then prove the relation: $\int U(n) dn = U(n) + 3\pi\sqrt{n}$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Tobi Joshua-Nigeria



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$$\begin{aligned}
 U(n) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1+x^2}{1+y^2} e^{-ny^2-x^2} dy dx = \\
 &= 4 \int_0^{\infty} (1+x^2) e^{-x^2} dx \int_0^{\infty} \frac{e^{-ny^2} dy}{1+y^2} \text{ (even function)} \\
 &= 2 \int_0^{\infty} \left(x^{-\frac{1}{2}} + x^{\frac{1}{2}} \right) e^{-x} dx \int_0^{\infty} \frac{e^{-ny^2} dy}{1+y^2} = \\
 &= 2 \left(\Gamma\left(\frac{1}{2}\right) + \Gamma\left(\frac{3}{2}\right) \right) \int_0^{\infty} e^{-ny^2} dy \int_0^{\infty} e^{-t(y^2+1)} dt = \\
 &= 3\sqrt{\pi} \int_0^{\infty} e^{-t} dt \int_0^{\infty} e^{-y^2(n+t)} dy = 3\sqrt{\pi} \int_0^{\infty} e^{-t} dt \frac{\sqrt{\pi}}{2\sqrt{n+t}} = 3\pi \int_0^{\infty} \frac{e^{-t}}{2\sqrt{n+t}} dt; \quad (1)
 \end{aligned}$$

$$\left(\text{since } \int_0^{\infty} e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \right)$$

$$\begin{aligned}
 \text{Now, } \int U(n) dn &= 3\pi \int_0^{\infty} \int \frac{e^{-t}}{2\sqrt{n+t}} dt dn = 3\pi \int_0^{\infty} \int e^{-t} (\sqrt{n+t} + C) dt \stackrel{IBP}{=} \\
 &= 3\pi \left[-e^{-t} \sqrt{n+t} \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-t}}{2\sqrt{n+t}} dt \right] = \\
 &= 3\pi \left[\sqrt{n} + \int_0^{\infty} \frac{e^{-t}}{2\sqrt{n+t}} dt \right] = 3\pi\sqrt{n} + 3\pi \int_0^{\infty} \frac{e^{-t}}{2\sqrt{n+t}} dt = 3\pi\sqrt{n} + U(n)
 \end{aligned}$$

1159. Find without any software:

$$\Omega = \int_0^{\pi} \frac{\sin^3 x}{9 - \cos^2 x} dx$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$\Omega = \int_0^{\pi} \frac{\sin^3 x}{9 - \cos^2 x} dx = \int_0^{\pi} \frac{\sin^3 x + 8\sin x - 8\sin x}{8 + \sin^2 x} dx =$$



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$$\begin{aligned}
 &= \int_0^\pi \frac{\sin x(8 + \sin^2 x)}{8 + \sin^2 x} dx - 8 \int_0^\pi \frac{\sin x}{8 + \sin^2 x} dx = \\
 &= \int_0^\pi \sin x dx - 8 \int_0^\pi \frac{\sin x}{9 - \cos^2 x} dx = 2 - 8 \int_0^\pi \frac{(\cos x)' dx}{\cos^2 x - 9} = \\
 &= 2 - \frac{8}{6} \left(\log \left| \frac{\cos \pi - 3}{\cos \pi + 3} \right| - \log \left| \frac{\cos 0 - 3}{\cos 0 + 3} \right| \right) = 2 - \frac{4}{3} \left(\log 2 - \log \frac{1}{2} \right) = 2 - \frac{4}{3} \log 4
 \end{aligned}$$

1160. If $0 < a \leq b$ then:

$$(b-a)^2 \int_a^b \frac{x^2 dx}{1+x^2} + (b-a) \int_a^b \int_a^b \frac{y^2 dxdy}{(1+x^2)(1+y^2)} + \int_a^b \int_a^b \int_a^b \frac{z^2 dx dy dz}{(1+x^2)(1+y^2)(1+z^2)} + \log^3 \left(\sqrt{\frac{b}{a}} \right) \geq (b-a)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Șerban-Romania

$$\begin{aligned}
 &(b-a)^2 \int_a^b \frac{x^2 dx}{1+x^2} + (b-a) \int_a^b \int_a^b \frac{y^2 dxdy}{(1+x^2)(1+y^2)} + \int_a^b \int_a^b \int_a^b \frac{z^2 dx dy dz}{(1+x^2)(1+y^2)(1+z^2)} + \log^3 \left(\sqrt{\frac{b}{a}} \right) \geq (b-a)^3 \\
 &(b-a)^2 \int_a^b \frac{(x^2+1)dx}{1+x^2} - (b-a)^2 \int_a^b \frac{dx}{1+x^2} + (b-a) \int_a^b \int_a^b \frac{(y^2+1)dxdy}{(1+x^2)(1+y^2)} - (b-a) \int_a^b \int_a^b \frac{dxdy}{(1+x^2)(1+y^2)} + \\
 &+ \int_a^b \int_a^b \int_a^b \frac{(z^2+1)dx dy dz}{(1+x^2)(1+y^2)(1+z^2)} - \int_a^b \int_a^b \int_a^b \frac{dx dy dz}{(1+x^2)(1+y^2)(1+z^2)} + \frac{1}{8} \log^3 \left(\frac{b}{a} \right) \geq (b-a)^3 \\
 &(b-a)^3 - (b-a)^2 \int_a^b \frac{dx}{1+x^2} + (b-a)^2 \int_a^b \frac{dx}{1+x^2} - (b-a) \int_a^b \int_a^b \frac{(y^2+1)dxdy}{(1+x^2)(1+y^2)} + \\
 &+ (b-a) \int_a^b \int_a^b \frac{(y^2+1)dxdy}{(1+x^2)(1+y^2)} - \int_a^b \int_a^b \int_a^b \frac{dx dy dz}{(1+x^2)(1+y^2)(1+z^2)} + \frac{1}{8} \log^3 \left(\frac{b}{a} \right) \geq (b-a)^3
 \end{aligned}$$

We must show:

$$\int_a^b \int_a^b \int_a^b \frac{dx dy dz}{(1+x^2)(1+y^2)(1+z^2)} \leq \frac{1}{8} \log^3 \left(\frac{b}{a} \right)$$

We have that:

$$\int_a^b \int_a^b \int_a^b \frac{dx dy dz}{(1+x^2)(1+y^2)(1+z^2)} \stackrel{AM-GM}{\leq} \int_a^b \int_a^b \int_a^b \frac{dx dy dz}{(2x)(2y)(2z)} = \frac{1}{8} \int_a^b \int_a^b \int_a^b \frac{dx dy dz}{xyz}$$



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$$\begin{aligned}
 &= \frac{1}{8} \int_a^b \int_a^b \frac{dxdy}{xy} \cdot \log z|_a^b = \\
 &= \frac{1}{8} \log\left(\frac{b}{a}\right) \int_a^b \frac{dx}{x} \cdot \log y|_a^b = \frac{1}{8} \log^2\left(\frac{a}{b}\right) \int_a^b \frac{dx}{x} = \frac{1}{8} \log^3\left(\frac{b}{a}\right) \text{ (true).}
 \end{aligned}$$

Solution 2 by Adrian Popa-Romania

$$1) \int_a^b \frac{x^2 dx}{1+x^2} = \int_a^b \frac{(x^2 + 1 - 1) dx}{1+x^2} = \int_a^b \left(1 - \frac{1}{1+x^2}\right) dx = (b-a) - (\tan^{-1} b - \tan^{-1} a)$$

$$\begin{aligned}
 2) \int_a^b \int_a^b \frac{y^2 dxdy}{(1+x^2)(1+y^2)} &= \int_a^b \frac{dx}{1+x^2} \int_a^b \frac{y^2 dy}{1+y^2} = (\tan^{-1} x|_a^b) \cdot (y|_a^b - \tan^{-1} y|_a^b) = \\
 &= (b-a)(\tan^{-1} b - \tan^{-1} a) - (\tan^{-1} b - \tan^{-1} a)^2
 \end{aligned}$$

$$\begin{aligned}
 3) \int_a^b \int_a^b \int_a^b \frac{dxdydz}{(1+x^2)(1+y^2)(1+z^2)} &= \int_a^b \frac{dx}{1+x^2} \int_a^b \frac{y^2 dy}{1+y^2} \int_a^b \frac{z^2 dz}{1+z^2} = \\
 &= (b-a)(\tan^{-1} b - \tan^{-1} a)^2 - (\tan^{-1} b - \tan^{-1} a)^3
 \end{aligned}$$

So,

$$\begin{aligned}
 S &= (b-a)^3 - (b-a)^2(\tan^{-1} b - \tan^{-1} a) + (b-a)^2(\tan^{-1} b - \tan^{-1} a) - \\
 &\quad -(b-a)(\tan^{-1} b - \tan^{-1} a)^2 + (b-a)(\tan^{-1} b - \tan^{-1} a)^2 - \\
 &\quad -(\tan^{-1} b - \tan^{-1} a)^3 + \log^3\left(\sqrt{\frac{b}{a}}\right) \geq (b-a)^3 \Leftrightarrow
 \end{aligned}$$

$$\log^3\left(\sqrt{\frac{b}{a}}\right) \geq (\tan^{-1} b - \tan^{-1} a)^3 \Leftrightarrow \frac{1}{2} \log\left(\frac{b}{a}\right) \geq \tan^{-1} b - \tan^{-1} a \Leftrightarrow$$

$$\frac{\log b - \log a}{\tan^{-1} b - \tan^{-1} a} \stackrel{(*)}{\geq} 2$$

Let be the functions: $f(x) = \log x$; $g(x) = \tan^{-1} x$, $x \in [a, b]$ and applying Cauchy

Theorem: $\exists c \in [a, b]$ such that:



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$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} = \frac{\frac{1}{c}}{\frac{1}{1+c^2}} = \frac{1+c^2}{c} \stackrel{AM-GM}{\geq} \frac{2c}{c} = 2 \Rightarrow (*) \text{true.}$$

1161. If $0 < a \leq b \leq \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \int_a^b (\tan x \tan y + 1)(\tan y \tan z + 1)(\tan z \tan x + 1) dx dy dz \leq (\tan b - \tan a)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Șerban-Romania

$$\begin{aligned} \prod_{cyc} (\tan x \tan y + 1) &= \prod_{cyc} \frac{\sin x \sin y + \cos x \cos y}{\cos x \cos y} = \frac{\prod_{cyc} \cos(x-y)}{\prod_{cyc} \cos^2 x} \leq \frac{1}{\prod_{cyc} \cos^2 x} \\ \int_a^b \int_a^b \int_a^b \prod_{cyc} (\tan x \tan y + 1) dx dy dz &\leq \left(\int_a^b \frac{1}{\cos^2 x} dx \right) \left(\int_a^b \frac{1}{\cos^2 y} dy \right) \left(\int_a^b \frac{1}{\cos^2 z} dz \right) \\ &= (\tan x|_a^b)^3 = (\tan b - \tan a)^3 \end{aligned}$$

Solution 2 by Remus Florin Stanca-Romania

$$\begin{aligned} \therefore (ab+1)^2 &\stackrel{CBS}{\leq} (a^2+1)(b^2+1) \\ (\tan x \tan y + 1)^2 &\leq (\tan^2 x + 1)(\tan^2 y + 1) \\ (\tan y \tan z + 1)^2 &\leq (\tan^2 y + 1)(\tan^2 z + 1) \\ (\tan z \tan x + 1)^2 &\leq (\tan^2 z + 1)(\tan^2 x + 1) \\ \stackrel{(\cdot)}{\Rightarrow} \left(\prod_{cyc} (\tan x \tan y + 1) \right)^2 &\leq \left(\prod_{cyc} (\tan^2 x + 1) \right)^2 \Rightarrow \\ \int_a^b \int_a^b \int_a^b \prod_{cyc} (\tan x \tan y + 1) dx dy dz &\leq \int_a^b \int_a^b \int_a^b \prod_{cyc} (\tan^2 x + 1) dx dy dz = \\ &= (\tan x|_a^b)^3 = (\tan b - \tan a)^3 \end{aligned}$$



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1162. If $\frac{2}{3} < a \leq b$ then:

$$\int_a^b x \cdot \sin \frac{\pi}{3x} dx \geq \sqrt{1+b^2} - \sqrt{1+a^2}$$

Proposed by Daniel Sitaru-Romania

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan

Lemma: If $x > 2$, then $\sin \frac{\pi}{x} > \frac{3}{\sqrt{x^2+9}}$

$$x > 2 \Rightarrow \pi x > 2\pi \Rightarrow \frac{\pi}{x} > \frac{\pi}{2} \text{ and } x < \tan x \Rightarrow \tan \frac{\pi}{x} > \frac{\pi}{x} > \frac{3}{x}; \quad (1)$$

Using $1 + \tan^2 x = \frac{1}{\cos^2 x}$ we have:

$$\cos^2 x = \frac{1}{1 + \tan^2 x} \stackrel{(1)}{<} \frac{1}{1 + \left(\frac{\pi}{x}\right)^2} \stackrel{(1)}{<} \frac{1}{1 + \left(\frac{3}{x}\right)^2} = \frac{x^2}{x^2 + 9}$$

$$\sin^2 x = 1 - \cos^2 x > 1 - \frac{x^2}{x^2 + 9} = \frac{9}{x^2 + 9}$$

$$\text{Now, take } x \rightarrow 3, \text{ then } \sin \frac{\pi}{3x} > \frac{3}{\sqrt{9x^2+9}} = \frac{1}{\sqrt{x^2+1}}; \quad (2)$$

$$\text{Hence, } \int_a^b x \cdot \sin \frac{\pi}{3x} dx \stackrel{(2)}{\geq} \int_a^b x \cdot \frac{1}{\sqrt{x^2+1}} dx$$

$$\int_a^b x \cdot \frac{1}{\sqrt{x^2+1}} dx = \frac{1}{2} \int_a^b \frac{d(x^2+1)}{\sqrt{x^2+1}} = \frac{1}{2} \cdot 2\sqrt{x^2+1} \Big|_a^b = \sqrt{1+b^2} - \sqrt{1+a^2}$$

$$\int_a^b x \cdot \sin \frac{\pi}{3x} dx \geq \sqrt{1+b^2} - \sqrt{1+a^2}$$

1163. If $f, f': (0, \infty) \rightarrow (0, \infty)$, f differentiable, $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{(f(x) + f(y))f'(x)f'(y)}{\sqrt{1+f(x)f(y)}} dx dy \leq \log \left(\frac{f(b) + \sqrt{1+f^2(b)}}{f(a) + \sqrt{1+f^2(a)}} \right)^{f^2(b)-f^2(a)}$$

Proposed by Daniel Sitaru-Romania

Solution by Remus Florin Stanca-Romania



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Let's prove that:

$$\frac{a+b}{\sqrt{1+ab}} \leq \frac{a}{\sqrt{1+b^2}} + \frac{b}{\sqrt{1+a^2}}, \forall a, b > 0$$

The inequality can be written as:

$$\frac{1}{\sqrt{1+ab}} \leq \frac{a}{a+b} \cdot \frac{1}{\sqrt{1+b^2}} + \frac{b}{a+b} \cdot \frac{1}{\sqrt{1+a^2}}$$

$$\text{Let } g: (0, \infty) \rightarrow (0, \infty), g(x) = \frac{1}{\sqrt{x+1}} \cdot \frac{\partial g}{\partial x} = -\frac{1}{2}(x+1)^{-\frac{3}{2}} \cdot \frac{\partial^2 g}{\partial x^2} = \frac{3}{2} \cdot \frac{1}{2}(x+1)^{-\frac{5}{2}} \geq 0$$

$\Rightarrow g$ –convexe, then for any $t_1, t_2 \in (0, 1)$, $t_1 + t_2 = 1$ and for any $x_1, x_2 \in I$ we have:

$$t_1 f(x_1) + t_2 f(x_2) \geq f(t_1 x_1 + t_2 x_2)$$

$$\text{Let } t_1 = \frac{a}{a+b}; t_2 = \frac{b}{a+b} \text{ and } x_1 = b^2, x_2 = a^2$$

$$\frac{a}{a+b} \cdot \frac{1}{\sqrt{1+b^2}} + \frac{b}{a+b} \cdot \frac{1}{\sqrt{1+a^2}} \geq \frac{1}{\sqrt{1+\frac{a^2b+ab^2}{a+b}}} = \frac{1}{\sqrt{1+\frac{ab(a+b)}{a+b}}} = \frac{1}{\sqrt{1+ab}}$$

$$\Rightarrow \frac{a+b}{\sqrt{1+ab}} \leq \frac{a}{\sqrt{1+b^2}} + \frac{b}{\sqrt{1+a^2}}, \forall a, b > 0$$

$$\frac{f(x) + f(y)}{\sqrt{1+f(x)f(y)}} \leq \frac{f(x)}{\sqrt{1+f^2(y)}} + \frac{f(y)}{\sqrt{1+f^2(x)}}$$

$$\frac{(f(x) + f(y))f'(x)f'(y)}{\sqrt{1+f(x)f(y)}} \leq \frac{f(x)f'(x)f'(y)}{\sqrt{1+f^2(y)}} + \frac{f(y)f'(x)f'(y)}{\sqrt{1+f^2(x)}}, \quad (3)$$

$$\frac{1}{2} \int_a^b \int_a^b \frac{2f(x)f'(x)f'(y)}{\sqrt{1+f^2(y)}} dx dy = \frac{1}{2} \int_a^b \left[\frac{f^2(x)f'(y)}{\sqrt{1+f^2(y)}} \right]_a^b dy =$$

$$= \frac{1}{2} (f^2(b) - f^2(a)) \left(\log(f(b) + \sqrt{1+f^2(b)}) - \log(f(a) + \sqrt{1+f^2(a)}) \right) =$$

$$= \frac{1}{2} (f^2(b) - f^2(a)) \log \left(\frac{f(b) + \sqrt{1+f^2(b)}}{f(a) + \sqrt{1+f^2(a)}} \right); \quad (1)$$

$$\frac{1}{2} \int_a^b \int_a^b \frac{2f(y)f'(y)f'(x)}{\sqrt{1+f^2(x)}} dx dy = \frac{1}{2} \int_a^b \left[\frac{f^2(y)f'(x)}{\sqrt{1+f^2(x)}} \right]_a^b dy =$$

$$= \frac{1}{2} (f^2(b) - f^2(a)) \left(\log(f(b) + \sqrt{1+f^2(b)}) - \log(f(a) + \sqrt{1+f^2(a)}) \right) =$$



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$$= \frac{1}{2} (f^2(b) - f^2(a)) \log \left(\frac{f(b) + \sqrt{1 + f^2(b)}}{f(a) + \sqrt{1 + f^2(a)}} \right); \quad (2)$$

From (1), (2), (3) it follows that:

$$\int_a^b \int_a^b \frac{(f(x) + f(y))f'(x)f'(y)}{\sqrt{1 + f(x)f(y)}} dx dy \leq \log \left(\frac{f(b) + \sqrt{1 + f^2(b)}}{f(a) + \sqrt{1 + f^2(a)}} \right)^{f^2(b) - f^2(a)}$$

1164. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$2 \int_a^b \int_a^b \cos^2 x \cos^2 y (1 + \tan x \tan y) |\tan x - \tan y| dx dy \leq (b - a)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaijan

$$\begin{aligned} & \cos^2 x \cos^2 y (1 + \tan x \tan y) |\tan x - \tan y| = \\ & = \cos^2 x \cos^2 y \left(1 + \frac{\sin x \sin y}{\cos x \cos y} \right) \left| \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y} \right| = \\ & = \cos(x - y) |\sin(x - y)| |\sin(x - y) \cos(x - y)| = \frac{1}{2} \sin 2(x - y) \end{aligned}$$

Therefore,

$$\begin{aligned} & 2 \int_a^b \int_a^b \cos^2 x \cos^2 y (1 + \tan x \tan y) |\tan x - \tan y| dx dy \leq \\ & \leq 2 \int_a^b \int_a^b \frac{1}{2} \sin 2(x - y) dx dy \leq \int_a^b \int_a^b dx dy = (b - a)^2 \end{aligned}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

Let us prove that:

$$\begin{aligned} & \cos^2 x \cos^2 y (1 + \tan x \tan y) |\tan x - \tan y| < \frac{1}{2} \Leftrightarrow \\ & \cos^2 x \cos^2 y (\tan x - \tan y)^2 < \frac{|\tan x - \tan y|}{2(1 + \tan x \tan y)} \end{aligned}$$



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Suppose $y \leq x$ we have:

$$\cos^2 x \cos^2 y (\tan x - \tan y)^2 < \frac{\tan x - \tan y}{2(1 + \tan x \tan y)} \Leftrightarrow$$

$$\cos^2 x \cos^2 y \cdot \frac{\sin^2(x-y)}{\cos^2 x \cos^2 y} \leq \frac{1}{2} \tan(x-y) \Leftrightarrow$$

$$\cos(x-y) \sin(x-y) \leq \frac{1}{2} \Leftrightarrow \sin 2(x-y) \leq 1 \text{ (true).}$$

Therefore,

$$\begin{aligned} 2 \int_a^b \int_a^b \cos^2 x \cos^2 y (1 + \tan x \tan y) |\tan x - \tan y| dx dy &\leq \\ &\leq 2 \int_a^b \int_a^b \frac{1}{2} \sin 2(x-y) dx dy \leq \int_a^b \int_a^b dx dy = (b-a)^2 \end{aligned}$$

1165. If $f: [0, 1] \rightarrow \mathbb{R}$ continuous function, $n \in \mathbb{N}$, $n \geq 1$ then prove:

$$e^{2n} + 2n \int_0^1 f^2(e^x) dx \geq 1 + 4n \int_1^e x^{n-1} f(x) dx$$

Proposed by Daniel Sitaru-Romania

Solution by Florică Anastase-Romania

$$\int_1^e x^{n-1} f(x) dx \stackrel{x=e^t \Rightarrow t=\log x}{=} \int_0^1 e^{nt} f(e^t) dt$$

Hence,

$$e^{2n} + 2n \int_0^1 f^2(e^x) dx \geq 1 + 4n \int_1^e x^{n-1} f(x) dx \Leftrightarrow$$

$$e^{2n} + 2n \int_0^1 f^2(e^x) dx \geq 1 + 4n \int_0^1 e^{nx} f(e^x) dx \Leftrightarrow$$



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$$\frac{e^{2n} - 1}{2n} - 2 \int_0^1 e^{nx} f(e^x) dx + \int_0^1 f^2(e^x) dx \geq 0 \Leftrightarrow$$

$$\int_0^1 e^{2nx} dx - 2 \int_0^1 e^{nx} f(e^x) dx + \int_0^1 f^2(e^x) dx \geq 0 \Leftrightarrow$$

$$\int_0^1 (e^{2nx} - 2e^{nx} f(e^x) + f^2(e^x)) dx \geq 0 \Leftrightarrow$$

$$\int_0^1 (e^{nx} - f(e^x))^2 dx \geq 0$$

1166. If $0 < a \leq b, f: (0, \infty) \rightarrow (0, \infty)$, f –continuous then:

$$\int_a^b f^6(x) dx \cdot \left(\int_a^b \frac{1}{f(x)} dx \right)^3 \geq (b-a) \left(\int_a^b f(x) dx \right)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

By Cauchy-Schwartz Inequality, we have:

$$\begin{aligned} \left(\int_a^b (f^3(x))^2 dx \right) \left(\int_a^b 1^2 dx \right) &\geq \left(\int_a^b f^3(x) \cdot 1 dx \right)^2 \Leftrightarrow \\ \left(\int_a^b (f^3(x))^2 dx \right) (b-a) &\geq \left(\int_a^b f^3(x) dx \right)^2 = \left(\int_a^b f^3(x) dx \right) \left(\int_a^b f^3(x) dx \right); (1) \end{aligned}$$

Other, by Holder's Inequality:

$$\left(\int_a^b f^3(x) dx \right)^{\frac{1}{3}} \left(\int_a^b 1^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \geq \int_a^b f(x) \cdot 1 dx \Leftrightarrow$$

$$\left(\int_a^b f^3(x) dx \right) (b-a) \geq \left(\int_a^b f(x) dx \right)^3; (2)$$



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And

$$\left(\int_a^b f(x) dx \right) \left(\int_a^b \frac{1}{f(x)} dx \right) \stackrel{BCS}{\geq} \left(\int_a^b f(x) \cdot \frac{1}{f(x)} dx \right)^2 = (b-a)^2; (3)$$

From (1),(2),(3) we have:

$$\begin{aligned} \int_a^b f^6(x) dx \cdot (b-a)^2 &\geq (b-a) \left(\int_a^b f^3(x) dx \right) \left(\int_a^b f^3(x) dx \right) \geq \\ &\geq \left(\int_a^b f(x) dx \right)^3 \left(\int_a^b f^3(x) dx \right) \Rightarrow \\ \int_a^b f^6(x) dx \cdot (b-a)^3 &\geq \left(\int_a^b f(x) dx \right)^3 (b-a) \int_a^b f^3(x) dx \\ &\geq \left(\int_a^b f(x) dx \right)^3 \left(\int_a^b f(x) dx \right)^3 \\ \int_a^b f^6(x) dx \cdot \left(\int_a^b \frac{1}{f(x)} dx \right)^3 (b-a)^3 &\geq \left(\int_a^b f(x) dx \right)^3 \left(\int_a^b f(x) dx \cdot \int_a^b \frac{1}{f(x)} dx \right)^3 \geq \\ &\geq \left(\int_a^b f(x) dx \right)^3 ((b-a)^2)^3 \\ \int_a^b f^6(x) dx \cdot \left(\int_a^b \frac{1}{f(x)} dx \right)^3 &\geq (b-a) \left(\int_a^b f(x) dx \right)^3 \end{aligned}$$

Solution 2 by Florică Anastase-Romania

Using Means Integral Inequality:

If $0 < a \leq b, f: (0, \infty) \rightarrow (0, \infty), f$ –continuous then:

$$\frac{b-a}{\int_a^b \frac{1}{f(x)} dx} \leq e^{\frac{1}{b-a} \int_a^b \log f(x) dx} \leq \frac{1}{b-a} \int_a^b f(x) dx$$



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We have:

$$\int_a^b \frac{1}{f(x)} dx \geq \frac{(b-a)^2}{\int_a^b f(x) dx} \Rightarrow \left(\int_a^b \frac{1}{f(x)} dx \right)^3 \geq \frac{(b-a)^6}{\left(\int_a^b f(x) dx \right)^3}; \quad (1)$$

From Bernoulli Integral Inequality:

If $0 < a \leq b, n > 1, f: (0, \infty) \rightarrow (0, \infty), f$ –continuous then:

$$\left(\int_a^b \frac{1}{f(x)} dx \right)^n \leq (b-a)^{n-1} \int_a^b f^n(x) dx$$

We have:

$$\int_a^b f^6(x) dx \geq \frac{1}{(b-a)^5} \left(\int_a^b f(x) dx \right)^6; \quad (2)$$

From (1),(2) we have:

$$\begin{aligned} \int_a^b f^6(x) dx \cdot \left(\int_a^b \frac{1}{f(x)} dx \right)^3 &\geq \frac{1}{(b-a)^5} \left(\int_a^b f(x) dx \right)^6 \frac{(b-a)^6}{\left(\int_a^b f(x) dx \right)^3} \Leftrightarrow \\ \int_a^b f^6(x) dx \cdot \left(\int_a^b \frac{1}{f(x)} dx \right)^3 &\geq (b-a) \left(\int_a^b f(x) dx \right)^3 \end{aligned}$$

1167. If $0 < a \leq b < \frac{\pi}{8080}$ then:

$$\int_a^b \frac{\tan(2021x) \cdot \tan(2022x) \cdot \tan(2023x)}{8\tan^3 x} dx \geq 10^9(b-a)$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

Let be the function $f(x) = \tan(nx) - ntanx, n \in \{2021, 2022, 2023\}, 0 < x < \frac{\pi}{8080}$

$$f'(x) = n(\tan(nx) - tanx)(\tan(nx) + tanx);$$

$$f'(x) = 0 \Rightarrow x = 0.$$

$f'(x) > 0$ and because $x \rightarrow \tan x$ –increasing, it follows that:



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$$\tan(nx) > \tan x, nx \in \left(0, \frac{1}{4} + \varepsilon\right), \varepsilon \rightarrow 0$$

$$f(0) = 0 \Rightarrow f(x) > 0 \Rightarrow \tan(nx) > n \tan x \Rightarrow \frac{\tan(x)}{\tan x} > n.$$

Therefore,

$$\begin{aligned} & \int_a^b \frac{\tan(2021x) \cdot \tan(2022x) \cdot \tan(2023x)}{8 \tan^3 x} dx \geq \\ & > \int_a^b \frac{2021 \cdot 2022 \cdot 2023}{8} dx \geq \int_a^b \frac{2000 \cdot 2000 \cdot 2000}{8} dx = 10^9(b-a) \end{aligned}$$

1168. If $0 < a \leq b$ then:

$$8 \int_a^b \int_a^b \frac{x^{12} + y^{12}}{x^5 + y^5} dx dy \geq (b-a)(b^8 - a^8)$$

Proposed by Daniel Sitaru-Romania

Solution by Oyebamiji Oluwaseyi-Nigeria

By Chebyshev's inequality:

$$(x^5 + y^5)(x^7 + y^7) \leq 2(x^{12} + y^{12})$$

Hence,

$$\begin{aligned} & 8 \int_a^b \int_a^b \frac{x^{12} + y^{12}}{x^5 + y^5} dx dy \geq 8 \int_a^b \int_a^b \frac{(x^5 + y^5)(x^7 + y^7)}{2(x^5 + y^5)} dx dy = \\ & = \frac{8}{2} \int_a^b \int_a^b (x^7 + y^7) dy dx = 4 \int_a^b \left((b-a)x^7 + \frac{b^8 - a^8}{8} \right) dx = \\ & = 4(b-a) \int_a^b x^7 dx + \frac{4(b^8 - a^8)}{8} \int_a^b dx = \frac{4(b-a)(b^8 - a^8)}{8} + \frac{4(b-a)(b^8 - a^8)}{8} = \\ & = (b-a)(b^8 - a^8) \end{aligned}$$



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1169. If $0 < a \leq b$ then prove:

$$(erf(b) - erf(a))^2 \leq \frac{16}{3} \left(erf\left(\frac{3a+b}{4}\right) - erf(a) \right) \left(erf\left(\frac{a+3b}{4}\right) - erf(a) \right)$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \text{ -concave and increasing for all } z > 0.$$

Denote $f(z) = erf(z)$ we have:

$$f\left(\frac{3a+b}{4}\right) = f\left(\frac{a+a+a+b}{4}\right) \stackrel{\text{Jensen}}{\geq} \frac{3f(a) + f(b)}{4}$$

Hence,

$$f\left(\frac{3a+b}{4}\right) - f(a) = \frac{3f(a) + f(b)}{4} - f(a) = \frac{f(b) - f(a)}{4}; \quad (1)$$

$$f\left(\frac{a+3b}{4}\right) = f\left(\frac{a+b+b+b}{4}\right) \stackrel{\text{Jensen}}{\geq} \frac{f(a) + 3f(b)}{4}$$

Hence,

$$f\left(\frac{a+3b}{4}\right) - f(a) \geq \frac{f(a) + 3f(b)}{4} - f(a) = \frac{3(f(b) - f(a))}{4}; \quad (2)$$

From (1),(2) we have:

$$\begin{aligned} \left(erf\left(\frac{3a+b}{4}\right) - erf(a) \right) \left(erf\left(\frac{a+3b}{4}\right) - erf(a) \right) &\geq \frac{f(b) - f(a)}{4} \cdot \frac{3(f(b) - f(a))}{4} = \\ &= \frac{3}{16} (erf(b) - erf(a)) \end{aligned}$$

Therefore,

$$(erf(b) - erf(a))^2 \leq \frac{16}{3} \left(erf\left(\frac{3a+b}{4}\right) - erf(a) \right) \left(erf\left(\frac{a+3b}{4}\right) - erf(a) \right)$$



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1170. If $0 < a \leq b, f: [a, b] \rightarrow (0, \infty), f$ –continuous, then:

$$(b-a) \left(\int_a^b f^3(x) dx \right) \left(\int_a^b \frac{dx}{f^2(x)} \right) \geq \left(\int_a^b \sqrt[3]{f^5(x)} dx \right) \left(\int_a^b \frac{dx}{\sqrt[3]{f(x)}} \right)^2$$

Proposed by Daniel Sitaru-Romania

Solution by Abdul Hannan-Tezpur-India

Let $g(x) = \sqrt[3]{f(x)}$. Then the desired inequality is equivalent to

$$(b-a) \left(\int_a^b g^9(x) dx \right) \left(\int_a^b \frac{dx}{g^6(x)} \right) \geq \left(\int_a^b g^5(x) dx \right) \left(\int_a^b \frac{dx}{g(x)} \right)^2$$

This is true, because

$$\begin{aligned} (b-a) \left(\int_a^b g^9(x) dx \right) \left(\int_a^b \frac{dx}{g^6(x)} \right) &\stackrel{\text{Chebyshev}}{\geq} \left(\int_a^b g^5(x) dx \right) \left(\int_a^b g^4(x) dx \right) \left(\int_a^b \frac{dx}{g^6(x)} \right) \\ &\stackrel{\text{CBS}}{\geq} \left(\int_a^b g^5(x) dx \right) \left(\int_a^b \frac{dx}{g(x)} \right)^2 \end{aligned}$$

1171. Prove without any software:

$$\int_0^\pi \frac{4\sin^3 x}{5 - \cos x} dx > 1$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by Rachid Iksi-Morocco

$$\begin{aligned} I &= \int_0^\pi \frac{4\sin^3 x}{5 - \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{4\sin^3 x}{5 - \cos x} dx + \int_{\frac{\pi}{2}}^\pi \frac{4\sin^3 x}{5 - \cos x} dx \\ x = \pi - t &\Rightarrow \int_{\frac{\pi}{2}}^\pi \frac{4\sin^3 x}{5 - \cos x} dx = - \int_{\frac{\pi}{2}}^0 \frac{4\sin^3 x}{5 + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{4\sin^3 x}{5 + \cos x} dx \end{aligned}$$



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$$I = \int_0^{\frac{\pi}{2}} 4\sin^3 x \left(\frac{1}{5 - \cos x} + \frac{1}{5 + \cos x} \right) dx = \int_0^{\frac{\pi}{2}} \frac{40\sin^3 x}{25 - \cos^2 x} dx$$

As $\sin x \geq 0, \forall x \in \left[0, \frac{\pi}{2}\right]$

$$25 - \cos^2 x < 25 \Rightarrow \frac{\sin^3 x}{25 - \cos^2 x} > \frac{\sin^3 x}{25} \Rightarrow I \geq \frac{40}{25} \int_0^{\frac{\pi}{2}} \sin^3 x dx$$

$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \int_0^{\frac{\pi}{2}} \sin x (1 - \cos^2 x) dx = \left(-\cos x + \frac{1}{3} \cos^3 x \right) \Big|_0^{\frac{\pi}{2}} = 1 - \frac{2}{3} = \frac{1}{3}$$

Therefore,

$$I = \int_0^{\pi} \frac{4\sin^3 x}{5 - \cos x} dx \geq \frac{40}{25} \cdot \frac{2}{3} = \frac{16}{15} > 1$$

Solution 2 by Ravi Prakash-New Delhi-India

$$I = \int_0^{\pi} \frac{4\sin^3 x}{5 - \cos x} dx = 4 \int_0^{\pi} \frac{(1 - \cos^2 x)\sin x}{5 - \cos x} dx$$

Put $5 - \cos x = t, \sin x dx = dt$

$$I = 4 \int_4^6 \frac{1 - (5 - t)^2}{t} dt = 4 \int_4^6 \left(10 - t - \frac{24}{t} \right) dt =$$

$$= 4 \left[-\frac{1}{2}(10 - t)^2 - 24 \log t \right] \Big|_4^6 = 4 \left(10 + 24 \log \frac{2}{3} \right)$$

$$I > 1 \Leftrightarrow 4 \left(10 + 24 \log \frac{2}{3} \right) > 1 \Leftrightarrow 24 \log \left(\frac{2}{3} \right) > -39 \Leftrightarrow \log \left(\frac{3}{2} \right) < \frac{13}{24}$$

As $\log(1 + x) < x - \frac{1}{2}x^2 + \frac{1}{3}x^3, 0 < x < 1$ hence,

$$\log \left(\frac{3}{2} \right) < \frac{10}{24} < \frac{13}{24}$$

Solution 3 by Nelson Javier Villaherrera Lopez-El Salvador

$$I = \int_0^{\pi} \frac{4\sin^3 x}{5 - \cos x} dx = 4 \int_0^{\pi} \frac{\sin^2 x \sin x}{5 - \cos x} dx = -4 \int_0^{\pi} \frac{(1 - \cos^2 x)(-\sin x)}{5 - \cos x} dx =$$

$$= -4 \int_1^{-1} \frac{1 - y^2}{5 - y} dy = 4 \int_0^1 \frac{1 - y^2}{5 - y} dy = 4 \int_{-1}^1 \frac{25 - 24 - y^2}{5 - y} dy =$$

$$= 4 \int_{-1}^1 \left(5 + y - \frac{24}{5 - y} \right) dy = 4 \left[5y + \frac{y^2}{2} + 24 \log(5 - y) \right] \Big|_{-1}^1 =$$



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$$= 8 \left[5 - 12 \log \left(\frac{3}{2} \right) \right] > 1$$

1172. If $0 < a \leq b$ then:

$$32 \int_a^b \int_a^b \int_a^b (x+y+z)^3 dx dy dz \geq 27(b^2-a^2)^3 + 108(b-a)(b^2-a^2)(b^3-a^3)$$

Proposed by Daniel Sitaru-Romania

Solution by Abdul Hannan-Tezpur-India

$$\begin{aligned}
 & \text{Lemma: } \forall x, y, z > 0, 4(x+y+z)^3 \geq 27(xyz + x^2y + y^2z + z^2x) \\
 \Rightarrow & 32 \int_a^b \int_a^b \int_a^b (x+y+z)^3 dx dy dz \geq 8 \int_a^b \int_a^b \int_a^b 27(xyz + x^2y + y^2z + z^2x) dx dy dz = \\
 & = 27 \int_a^b \int_a^b \int_a^b 8xyz dx dy dz + 36 \sum_{cyc} \int_a^b \int_a^b \int_a^b 6x^2y dx dy dz = \\
 & = 27 \left(\int_a^b 2x dx \right) \left(\int_a^b 2y dy \right) \left(\int_a^b 2z dz \right) + 36 \sum_{cyc} \left(\int_a^b 3x^2 dx \right) \left(\int_a^b 2y dy \right) \left(\int_a^b dz \right) = \\
 & = 27(b^2-a^2)(b^2-a^2)(b^2-a^2) + 36 \sum_{cyc} (b^3-a^3)(b^2-a^2)(b-a) = \\
 & = 27(b^2-a^2)^3 + 108(b-a)(b^2-a^2)(b^3-a^3)
 \end{aligned}$$

Proof of lemma (by editor):

$$\begin{aligned}
 & \text{WLOG: } x \leq y \leq z, y = x+a, z = x+b, a, b \geq 0 \\
 & 4(x+y+z)^3 \geq 27(xyz + x^2y + y^2z + z^2x) \leftrightarrow \\
 & 4(3x+a+b)^3 \geq 27x(x+a)(x+b) + 27x((x+a)^2 + (x+b)^2 + x(x+a)) \\
 & 4(27x^3 + (a+b)^3 + 9x(a+b)(3x+a+b)) \\
 & \geq 27x(x^2 + 2ax + a^2 + x^2 + 2bx + b^2 + x^2 + ax + x^2 + ax + bx + ab) \\
 & 4(a+b)^3 + 36x(3ax + 3bx + a^2 + 2ab + b^2) \geq 27x(4ax + 3bx + a^2 + b^2 + ab) \\
 & 4(a+b)^3 + 9x(12ax + 12bx + 4a^2 + 8ab + 4b^2 - 12ax - 9bx - 3a^2 - 3b^2 - 3ab) \geq 0 \\
 & 4(a+b)^3 + 9x(3bx + a^2 + b^2 + 5ab) \geq 0 \\
 & \text{Equality holds for } a = b = 0 \leftrightarrow x = y = z.
 \end{aligned}$$



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1173. If $0 < a \leq b, f: (0, \infty) \rightarrow (0, \infty)$, f –continuous then;

$$\int_a^b \int_a^b \int_a^b \frac{2dxdydz}{f(x)f(y) + f^2(z)} \leq (b-a) \left(\int_a^b \frac{dx}{f(x)} \right)^2$$

Proposed by Daniel Sitaru-Romania

Solution by Ali Jaffal-Lebanon

Let $a \leq x, y, z \leq b$ then by AGM inequality we get:

$$f(x)f(y) + f^2(z) \geq 2\sqrt{f(x)f(y)f^2(z)} \geq 2f(z)\sqrt{f(x)f(y)}$$

Hence,

$$\begin{aligned} \frac{2}{f(x)f(y) + f^2(z)} &\leq \frac{1}{f(z)\sqrt{f(x)f(y)}} \\ \int_a^b \int_a^b \int_a^b \frac{2dxdydz}{f(x)f(y) + f^2(z)} &\leq \int_a^b \int_a^b \int_a^b \frac{dxdydz}{f(z)\sqrt{f(x)f(y)}} \leq \\ &\leq \int_a^b \frac{dx}{\sqrt{f(x)}} \cdot \int_a^b \frac{dy}{\sqrt{f(y)}} \cdot \int_a^b \frac{dz}{f(z)} = \left(\int_a^b \frac{dx}{\sqrt{f(x)}} \right)^2 \cdot \int_a^b \frac{dx}{f(x)} \end{aligned}$$

By Cauchy-B-S inequality we have:

$$\left(\int_a^b \frac{dx}{\sqrt{f(x)}} \right)^2 = \left(\int_a^b \mathbf{1} \cdot \frac{dx}{\sqrt{f(x)}} \right)^2 \leq \int_a^b \mathbf{1} dx \cdot \int_a^b \frac{dx}{f(x)} \leq (b-a) \int_a^b \frac{dx}{f(x)}$$

Therefore,

$$\begin{aligned} \int_a^b \int_a^b \int_a^b \frac{2dxdydz}{f(x)f(y) + f^2(z)} &\leq (b-a) \left(\int_a^b \frac{dx}{f(x)} \right) \left(\int_a^b \frac{dx}{f(x)} \right) \\ \int_a^b \int_a^b \int_a^b \frac{2dxdydz}{f(x)f(y) + f^2(z)} &\leq (b-a) \left(\int_a^b \frac{dx}{f(x)} \right)^2 \end{aligned}$$

1174. If $0 < a \leq b$; $f, f': (0, \infty) \rightarrow (0, \infty)$, f –derivable, f' –continuous,
 then:

$$18 \int_a^b \int_a^b \int_a^b \frac{f(x)f'(y)f'(z)dx dy dz}{(f(y) + 2f(z))(3f^2(x) + 2f^2(y) + f^2(z))} \leq (b-a)\log^2\left(\frac{f(b)}{f(a)}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

We show that:

$$\begin{aligned} \frac{18f(x)f'(y)f'(z)}{(f(y) + 2f(z))(3f^2(x) + 2f^2(y) + f^2(z))} &\leq \frac{f'(y)f'(z)}{f(y)f(z)} \\ \Leftrightarrow 18f(x)f(y)f(z) &\leq \\ \leq 3f^2(x)f(y) + 2f^3(y) + f(y)f^2(z) + 6f(z) + f^2(x) + 4f(z)f^2(y) + 2f^2(z) & \end{aligned}$$

Now,

$$\begin{aligned} 3f^2(x)f(y) + 2f^3(y) + f(y)f^2(z) + 6f(z) + f^2(x) + 4f(z)f^2(y) + 2f^2(z) &\stackrel{AGM}{\geq} \\ \stackrel{AGM}{\geq} 18\sqrt[18]{f^{18}(x)f^{18}(y)f^{18}(z)} &= 18f(x)f(y)f(z) \end{aligned}$$

Therefore,

$$18 \int_a^b \int_a^b \int_a^b \frac{f(x)f'(y)f'(z)dx dy dz}{(f(y) + 2f(z))(3f^2(x) + 2f^2(y) + f^2(z))} \leq (b-a)\log^2\left(\frac{f(b)}{f(a)}\right)$$

1175. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \lim_{x \rightarrow 0} \frac{1 - \sqrt[3]{\cos x} \cdot \sqrt[5]{\cos 3x} \cdot \dots \cdot \sqrt[2n+1]{\cos(2n-1)x}}{x^2} \right)$$

Proposed by Costel Florea-Romania

Solution 1 by Ravi Prakash-New Delhi-India

We shall use $\cos \theta = 1 - \frac{1}{2!}\theta^2 + o(\theta^4)$

$$\lim_{x \rightarrow 0} \frac{1}{x^2} \left[1 - (\cos x)^{\frac{1}{3}} \cdot (\cos 3x)^{\frac{1}{5}} \cdot \dots \cdot (\cos(2n-1)x)^{\frac{1}{2n+1}} \right] =$$



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$$= \lim_{x \rightarrow 0} \frac{1}{x^2} \left[1 - \left(1 - \frac{1}{2}x^2 + o(x^4) \right)^{\frac{1}{3}} \cdot \left(1 - \frac{9}{2}x^2 + o(x^4) \right)^{\frac{1}{5}} \cdots \cdot \left(1 - \frac{(2n-1)^2}{2}x^2 + o(x^4) \right)^{\frac{1}{2n+1}} \right] =$$

$$= \lim_{x \rightarrow 0} \frac{1}{x^2} \left[1 - \left(1 - \frac{1}{2}x^2 \left\{ \frac{1}{3} + \frac{3^2}{5} + \frac{5^2}{7} + \cdots + \frac{(2n-1)^2}{2n+1} \right\} + o(x^4) \right) \right] = \\ = \frac{1}{2} \left(\frac{1}{3} + \frac{3^2}{5} + \frac{5^2}{7} + \cdots + \frac{(2n-1)^2}{2n+1} \right)$$

$$\text{Let } t_r = \frac{(2r-1)^2}{2(2r+1)} = \frac{(2r+1-2)^2}{2(2r+1)} = \frac{2r-1}{2} - 2 + \frac{2}{2r+1} = \frac{1}{2}(2r-1) - 1 + \frac{2}{2r+1}$$

$$\sum_{r=1}^n t_r = \frac{1}{2} n^2 - n + 2b_n, \text{ where } b_n = \sum_{r=1}^n \frac{1}{2r+1}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \cdot \sum_{r=1}^n \frac{1}{2r+1} = \frac{1}{2}$$

$$\left(\because 0 < b_n < n \Rightarrow \lim_{n \rightarrow \infty} \frac{b_n}{n^2} = 0 \right)$$

Solution 2 by Rovsen Pirguliev-Azerbaijan

Denote $a_n = \lim_{x \rightarrow 0} \frac{1 - \sqrt[3]{\cos x} \cdot \sqrt[5]{\cos 3x} \cdots \sqrt[2n+1]{\cos(2n-1)x}}{x^2}$ then

$$a_n - a_{n-1} =$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt[3]{\cos x} \cdot \sqrt[5]{\cos 3x} \cdots \sqrt[2n-1]{\cos(2n-3)x} \cdot \left(1 - \sqrt[2n+1]{\cos(2n-1)x} \right)}{x^2} =$$

$$= \lim_{x \rightarrow 0} \sqrt[3]{\cos x} \cdot \sqrt[5]{\cos 3x} \cdots \sqrt[2n-1]{\cos(2n-3)x} \cdot \lim_{x \rightarrow 0} \frac{1 - \sqrt[2n+1]{\cos(2n-1)x}}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{1 - \sqrt[2n+1]{\cos(2n-1)x}}{x^2} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{\frac{2n-1}{2n+1} \cdot \frac{\sin(2n-1)x}{2x}}{\sqrt[2n+1]{(\cos(2n-1)x)^{2n}}} =$$

$$= \frac{1}{2} \cdot \frac{(2n-1)^2}{2n+1} = \frac{(2n-1)^2}{4n+2}$$

$$a_n - a_{n-1} = \frac{(2n-1)^2}{4n+2} = n - \frac{6n-1}{4n+2}; \quad (1)$$



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$$\left\{ \begin{array}{l} a_2 - a_1 = 2 - \frac{11}{10} \\ \vdots \\ a_n - a_{n-1} = n - \frac{6n-1}{4n+2} \end{array} \right. \xrightarrow{(+) a_n = 1 + 2 + 3 + \dots + n - A(n)} a_n = \frac{(2+n)(n-1)}{2} - A(n); (2)$$

So, we have:

$$\Omega = \lim_{n \rightarrow \infty} \frac{a_n}{n^2} \stackrel{(2)}{=} \lim_{n \rightarrow \infty} \frac{\frac{(2+n)(n-1)}{2} - A(n)}{n^2} = \frac{1}{2}$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$\begin{aligned} \lim_{x \rightarrow 0} g(x) &= \lim_{x \rightarrow 0} \frac{1 - \sqrt[3]{\cos x} \cdot \sqrt[5]{\cos 3x} \cdot \dots \cdot \sqrt[2n+1]{\cos(2n-1)x}}{x^2} = \\ &= - \lim_{x \rightarrow 0} \frac{\left(1 - \sqrt[3]{\cos x} \cdot \sqrt[5]{\cos 3x} \cdot \dots \cdot \sqrt[2n+1]{\cos(2n-1)x}\right)'}{2x} = \\ &= - \frac{1}{2} \lim_{x \rightarrow 0} \frac{\left(\sqrt[3]{\cos x} \cdot \sqrt[5]{\cos 3x} \cdot \dots \cdot \sqrt[2n+1]{\cos(2n-1)x}\right)'}{x} = \\ &= - \frac{1}{2} \lim_{x \rightarrow 0} \frac{(\sqrt[3]{\cos x})' \cdot (\sqrt[5]{\cos 3x})' \cdot \dots \cdot (\sqrt[2n+1]{\cos(2n-1)x})'}{x} \end{aligned}$$

$$\begin{aligned} \text{Hence, } \lim_{x \rightarrow 0} g(x) &= \frac{1}{2} \left(\frac{1}{3} + \frac{3^2}{5} + \frac{5^2}{7} + \dots + \frac{(2n-1)^2}{2n+1} \right) = \\ &= \frac{1}{2} \left(-1 + \frac{4}{3} + 1 + \frac{4}{5} + 3 + \frac{4}{7} + \dots + 2n-3 + \frac{4}{2n+1} \right) = \\ &= \frac{1}{2} \left(\frac{4}{3} + \frac{4}{5} + \frac{4}{7} + \dots + \frac{4}{2n+1} + 3 + 5 + \dots + 2n-3 \right) \end{aligned}$$

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left[\frac{1}{2n^2} \left(\frac{4}{3} + \frac{4}{5} + \frac{4}{7} + \dots + \frac{4}{2n+1} + 3 + 5 + \dots + 2n-3 \right) \right] = \\ &= \lim_{n \rightarrow \infty} \frac{(n-2) \cdot \frac{3+2n-3}{2}}{2n^2} = \lim_{n \rightarrow \infty} \frac{n(n-2)}{2n^2} = \frac{1}{2} \end{aligned}$$

1176. Find:

$$\Omega = \lim_{n \rightarrow \infty} n^{m+2} \cdot \int_0^1 \left(\frac{1}{(n - \sqrt[m]{x})^{m+1}} - \frac{1}{(n + \sqrt[m]{x})^{m+1}} \right) dx; \quad m \in \mathbb{N}^*$$

Proposed by Mohamed Bouras-Morocco



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Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

Note that:

$$B(u, v) = p^u q^v \int_0^1 \frac{x^{u-1} (1-x)^{v-1}}{(q + (p-q)x)^{u+v}} dx ; \quad (1)$$

Now,

$$\begin{aligned} \text{Let: } \varphi(n) &= n^{m+2} \cdot \int_0^1 \left(\frac{1}{(n - \sqrt[m]{x})^{m+1}} - \frac{1}{(n + \sqrt[m]{x})^{m+1}} \right) dx \stackrel{x \leftrightarrow x^n}{=} \\ &= m \cdot n^{m+2} \int_0^1 \left(\frac{x^{m-1}}{(n - x)^{m+1}} - \frac{x^{m-1}}{(n + x)^{m+1}} \right) dx \stackrel{(1)}{=} \\ &\stackrel{(1)}{=} m \cdot n^{m+2} \left(\frac{B(m, 1)}{n(n-1)^m} - \frac{B(m, 1)}{n(n+1)^m} \right) = n^{m+1} \left(\frac{(n+1)^m - (n-1)^m}{(n^2-1)^m} \right) = \\ &= n^{m+1} \cdot \frac{((n+1) - (n-1))((n+1)^{m-1} + (n+1)^{m-2}(n-1) + \dots + (n-1)^{m-1})}{(n^2-1)^m} \\ \Omega &= \lim_{n \rightarrow \infty} n^{m+2} \cdot \int_0^1 \left(\frac{1}{(n - \sqrt[m]{x})^{m+1}} - \frac{1}{(n + \sqrt[m]{x})^{m+1}} \right) dx = \lim_{n \rightarrow \infty} \varphi(n) = 2m \end{aligned}$$

Solution 2 by Kamel Benaicha-Algiers-Algerie

$$\begin{aligned} \Omega(m, n) &= \int_0^1 \left(\frac{1}{(n - \sqrt[m]{x})^{m+1}} - \frac{1}{(n + \sqrt[m]{x})^{m+1}} \right) dx, m \in \mathbb{N}^* \\ \Omega(m, n) &\stackrel{t=\sqrt[m]{x}}{=} m \int_0^1 \left(\frac{t^{m-1}}{(n - t)^{m+1}} - \frac{t^{m-1}}{(n + t)^{m+1}} \right) dt \\ \Omega_1(m, n) &= m \int_0^1 \frac{t^{m-1}}{(n - t)^{m+1}} dt = \frac{1}{(n-1)^m} - (m-1) \int_0^1 \frac{t^{m-2}}{(n-t)^m} dt \\ \Omega_1(m, n) &= \frac{1}{(n-1)^m} - \Omega_1(m-1, n) \\ \Omega_1(m, n) &= \frac{1}{(n-1)^m} - \frac{1}{(n-1)^{m-1}} - \Omega_1(m-2, n) = \end{aligned}$$



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$$= \sum_{p=0}^{m-2} (-1)^p \left(\frac{1}{(n-1)^{m-p}} + \frac{1}{(n+1)^{m-p}} \right) + (-1)^m \Omega(\mathbf{1}, n)$$

$$= \sum_{p=0}^{m-1} (-1)^p \cdot \frac{1}{(n-1)^{m-p}} + (-1)^{m-1} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$$\Omega_2(m, n) = m \int_0^1 \frac{t^{m-1}}{(n+t)^{m+1}} dt = \frac{1}{(n+1)^m} + (m-1) \int_0^1 \frac{t^{m-2}}{(n+t)^m} dt$$

$$\Omega_1(m, n) = \frac{1}{(n+1)^m} + \Omega_1(m-1, n) = \sum_{p=0}^{m-1} \frac{1}{(n+1)^{m-p}} - \frac{1}{n} =$$

$$= \sum_{p=0}^{m-1} \left(\frac{1}{(n+1)^{m-p}} + \frac{(-1)^p}{(n-1)^{m-p}} \right) + \frac{(-1)^m - 1}{n} =$$

$$= \frac{1}{(n+1)^m} \cdot \frac{(n+1)^m - 1}{n} - \frac{1}{(n-1)^m} \cdot \frac{(1-n)^m - 1}{n} + \frac{(-1)^m - 1}{n}$$

If $m = 2p$ we have:

$$\Omega(m, n) = \frac{1}{(n+1)^m} \cdot \frac{(n+1)^m - 1}{n} - \frac{1}{(n-1)^m} \cdot \frac{(1-n)^m - 1}{n}$$

$$\lim_{n \rightarrow \infty} n^{m+2} \cdot \Omega(m, n) = \lim_{n \rightarrow \infty} n^{m+1} \cdot \left(\frac{(n+1)^m - 1}{(n+1)^m} - \frac{(1-n)^m - 1}{(n-1)^m} \right) =$$

$$= \lim_{n \rightarrow \infty} n \left(\frac{1}{\left(1 - \frac{1}{n}\right)^m} - \frac{1}{\left(1 + \frac{1}{n}\right)^m} \right) \stackrel{x=\frac{1}{n}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{(1-x)^m} - \frac{1}{(1+x)^m}}{x} = m + m = 2m$$

If $m = 2p + 1$ we have:

$$\Omega(m, n) = \frac{1}{(n+1)^m} \cdot \frac{(n+1)^m - 1}{n} - \frac{1}{(n-1)^m} \cdot \frac{(1-n)^m - 1}{n} - \frac{2}{n}$$

$$\lim_{n \rightarrow \infty} n^{m+2} \cdot \Omega(m, n) = \lim_{n \rightarrow \infty} n^{m+1} \cdot \left(\frac{(n+1)^m - 1}{(n+1)^m} - \frac{(1-n)^m - 1}{(n-1)^m} - 2 \right) =$$

$$= \lim_{n \rightarrow \infty} n \left(\frac{1}{\left(1 - \frac{1}{n}\right)^m} - \frac{1}{\left(1 + \frac{1}{n}\right)^m} \right) \stackrel{x=\frac{1}{n}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{(1-x)^m} - \frac{1}{(1+x)^m}}{x} = m + m = 2m$$



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$$\text{So: } \lim_{n \rightarrow \infty} n^{m+2} \cdot \Omega(m, n) = 2m$$

$$\Omega = \lim_{n \rightarrow \infty} n^{m+2} \cdot \int_0^1 \left(\frac{1}{(n - \sqrt[m]{x})^{m+1}} - \frac{1}{(n + \sqrt[m]{x})^{m+1}} \right) dx$$

1177. Find:

$$\Omega = \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florică Anastase-Romania

$$\text{Let } \Omega_1 = \prod_{k=1}^n \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) \Rightarrow$$

$$\log \Omega_1 = \log \left[\prod_{k=1}^n \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) \right] = \sum_{k=1}^n \log \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right); \quad (1)$$

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1 \Rightarrow \forall \varepsilon > 0: 1 - \varepsilon \leq \frac{\log(1+x)}{x} \leq 1 + \varepsilon \Leftrightarrow$$

$$(1 - \varepsilon) \sum_{k=1}^n \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \leq \log \prod_{k=1}^n \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) \leq (1 + \varepsilon) \sum_{k=1}^n \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \Leftrightarrow$$

$$(1 - \varepsilon) \frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} \leq \sum_{k=1}^n \log \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) \leq (1 + \varepsilon) \frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2}; \quad (2)$$

$$f: [0, 1] \rightarrow \mathbb{R}, f(x) = \frac{\tan^{-1} x}{1 + x^2}, x_n^k = \xi_n^k = \frac{k}{n}, |\Delta_n| = \frac{1}{n} \rightarrow 0$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} = \lim_{n \rightarrow \infty} \sum_{k=1}^n f \left(\frac{k}{n} \right) \left(\frac{k}{n} - \frac{k-1}{n} \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(\xi_n^k)(x_n^k - x_n^{k-1}) =$$

$$= \int_0^1 f(x) dx = \int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx = \frac{1}{2} (\tan^{-1} x)^2 \Big|_0^1 = \frac{1}{2} ((\tan^{-1} 1)^2 - (\tan^{-1} 0)^2) = \frac{\pi^2}{32}; \quad (3)$$



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From (2),(3) we have:

$$\log \Omega_1 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \log \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) = \log \left(\frac{\pi^2}{32} \right)$$

$$\Omega = \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) = e^{\frac{\pi^2}{32}}$$

Solution 2 by Ali Jaffal-Lebanon

$$\text{We have: } x - \frac{x^2}{2} \leq \log(1 + x) \leq x, \forall x \geq 0; \quad (*)$$

$$\text{Let: } S_n = \sum_{k=1}^n \log \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) \text{ and } P_n = \prod_{k=1}^n \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right)$$

By (*) we have:

$$\frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} - \frac{1}{2} \cdot \frac{n^2 \left(\tan^{-1} \left(\frac{k}{n} \right) \right)^2}{(k^2 + n^2)^2} \leq \log \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) \leq \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2}$$

$$\frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} - \frac{1}{2} \left(\frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} \right)^2 \leq S_n \leq \frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} &= \int_0^1 f(x) dx = \int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx = \frac{1}{2} (\tan^{-1} x)^2 \Big|_0^1 \\ &= \frac{1}{2} ((\tan^{-1} 1)^2 - (\tan^{-1} 0)^2) = \frac{\pi^2}{32} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} \right)^2 = 0 \cdot \left(\int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx \right)^2 = 0$$

$$\lim_{n \rightarrow \infty} S_n = \frac{\pi^2}{32} \Rightarrow \lim_{n \rightarrow \infty} P_n = e^{\frac{\pi^2}{32}}$$

Solution 3 by Toby Joshua-Nigeria



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$$\Omega_1 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \log \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \log \left(1 + \frac{1}{n} \cdot \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} \right)$$

$$\log \left(1 + \frac{1}{n} \cdot \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} \right) \sim o \left(\frac{1}{n} \cdot \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} \right), \forall \left(\frac{k}{n} \right) \geq 0$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \cdot \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} & \stackrel{x=\frac{k}{n}}{=} \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx = \int_0^1 \tan^{-1} x d(\tan^{-1} x) dx = \\ & = \frac{1}{2} (\tan^{-1} x)^2 \Big|_0^1 = \frac{\pi^2}{32} \end{aligned}$$

$$\Omega = \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) = e^{\frac{\pi^2}{32}}$$

1178. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(n \left(\sqrt[n]{4} - 1 \right) - 2 \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} \right)^{2n}$$

Proposed by H.Tarverdi-Baku-Azerbaijan

Solution by Kamel Benaicha-Algiers-Algerie

$$\Omega = \lim_{n \rightarrow \infty} \left(n \left(\sqrt[n]{4} - 1 \right) - 2 \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} \right)^{2n} = \lim_{n \rightarrow \infty} (1 + n \left(\sqrt[n]{4} - 1 \right) - 2 \log 2)^{2n} =$$

$$= e^{\lim_{n \rightarrow \infty} 2n \log(1 + n(2^{\frac{2}{n}} - 1 - 2 \log 2))} \stackrel{x=\frac{1}{n}}{=} e^{\lim_{x \rightarrow 0} \frac{2 \log \left(1 + \frac{2^{2x}-1}{x} - 2 \log 2 \right)}{x}} =$$

$$= e^{\lim_{x \rightarrow 0} \frac{2 \left(\frac{2^{2x}-1}{x} - 2 \log 2 \right)}{x}} = e^{\lim_{x \rightarrow 0} \frac{2(2x \cdot 2^x \log 2 - 2^x + 1)}{x^2}} \stackrel{L'H}{=}$$

$$= e^{\lim_{x \rightarrow 0} \frac{4x \cdot 2^x \log^2 2 + 2 \cdot 2^x \log 2 - 2 \cdot 2^x \log 2}{x}} = e^{4 \log^2 2} = 4^{\log 4}$$

$$\Omega = \lim_{n \rightarrow \infty} \left(n \left(\sqrt[n]{4} - 1 \right) - 2 \sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n} \right)^{2n} = 4^{\log 4}$$

1179. If $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are positive real sequences defined by

$$a_n = \sum_{k=1}^n \frac{1}{k} \text{ and } \frac{b_{n+3}}{b_n} \left(\frac{b_{n+1}}{b_{n+2}} \right)^3 = \left(\frac{e^{a_n}}{n} \right)^3, \forall n \in \mathbb{N}^*$$

$$\text{Then find: } \Omega = \lim_{n \rightarrow \infty} \sqrt[n^3]{b_n}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Marian Ursărescu-Romania

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \sqrt[n^3]{b_n} \Rightarrow \log \Omega = \lim_{n \rightarrow \infty} \log \left(\sqrt[n^3]{b_n} \right) = \lim_{n \rightarrow \infty} \frac{\log b_n}{n^3} \stackrel{L.C-S}{=} \lim_{n \rightarrow \infty} \frac{\log b_{n+1} - \log b_n}{(n+1)^3 - n^3} = \\ &= \lim_{n \rightarrow \infty} \frac{\log \left(\frac{b_{n+1}}{b_n} \right)}{3n^2 + 3n + 1} = \lim_{n \rightarrow \infty} \frac{n^2}{3n^2 + 3n + 1} \cdot \frac{\log \left(\frac{b_{n+1}}{b_n} \right)}{n^2} \stackrel{L.C-S}{=} \\ &= \frac{1}{3} \cdot \lim_{n \rightarrow \infty} \frac{\log \left(\frac{b_{n+2}}{b_{n+1}} \right) - \log \left(\frac{b_{n+1}}{b_n} \right)}{(n+1)^2 - n^2} = \frac{1}{3} \cdot \lim_{n \rightarrow \infty} \frac{\log \left(\frac{b_n \cdot b_{n+2}}{b_{n+1}^2} \right)}{2n+1} = \\ &= \frac{1}{3} \cdot \lim_{n \rightarrow \infty} \frac{n}{2n+1} \cdot \frac{\log \left(\frac{b_n \cdot b_{n+2}}{b_{n+1}^2} \right)}{n} \stackrel{L.C-S}{=} \frac{1}{6} \cdot \lim_{n \rightarrow \infty} \frac{\log \left(\frac{b_{n+1} \cdot b_{n+3}}{b_{n+2}^2} \right) - \log \left(\frac{b_n \cdot b_{n+2}}{b_{n+1}^2} \right)}{n+1-n} = \\ &= \frac{1}{6} \cdot \lim_{n \rightarrow \infty} \log \left(\frac{b_{n+3}}{b_n} \cdot \frac{b_{n+1}^3}{b_{n+2}^3} \right) = \frac{1}{6} \cdot \lim_{n \rightarrow \infty} \log \left(\frac{e^{a_n}}{n} \right)^3 = \frac{1}{2} \cdot \lim_{n \rightarrow \infty} \log \left(\frac{e^{a_n}}{e^{\log n}} \right) = \\ &= \frac{1}{2} \cdot \lim_{n \rightarrow \infty} \log e^{a_n - \log n} = \frac{1}{2} \cdot e^{\log \gamma} = \frac{\gamma}{2} \Rightarrow \log \Omega = \frac{\gamma}{2} \Rightarrow \\ &\Omega = \lim_{n \rightarrow \infty} \sqrt[n^3]{b_n} = \sqrt{e^\gamma} \end{aligned}$$

1180. Find:

$$\lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a^n + b} + \sqrt[n]{a + b^n}}{a + b} \right)^{na^n b^n}; a, b > 1$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Adrian Popa-Romania



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$$\begin{aligned}
 L &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a^n + b} + \sqrt[n]{a + b^n}}{a + b} \right)^{na^n b^n} = \lim_{n \rightarrow \infty} \left(\frac{a \sqrt[n]{1 + \frac{b}{a^n}} + b \sqrt[n]{\frac{a}{b^n} + 1}}{a + b} \right)^{na^n b^n} = \\
 &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{a \sqrt[n]{1 + \frac{b}{a^n}} - a + b \sqrt[n]{\frac{a}{b^n} + 1} - b}{a + b} \right)^{\frac{1}{M}} \right]^{Mna^n b^n} = e^{\lim_{n \rightarrow \infty} Mna^n b^n} \\
 L_1 &= \lim_{n \rightarrow \infty} Mna^n b^n = \lim_{n \rightarrow \infty} \left(\frac{a \sqrt[n]{1 + \frac{b}{a^n}} - a + b \sqrt[n]{\frac{a}{b^n} + 1} - b}{a + b} \right) na^n b^n = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{a \left(\sqrt[n]{1 + \frac{b}{a^n}} - 1 \right) + b \left(\sqrt[n]{\frac{a}{b^n} + 1} - 1 \right)}{a + b} \right) na^n b^n = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{\frac{ab}{a^n}}{\sqrt[n]{\left(1 + \frac{b}{a^n}\right)^{n-1}} + \sqrt[n]{\left(1 + \frac{b}{a^n}\right)^{n-2} + \dots + 1}} + \frac{\frac{ab}{b^n}}{\sqrt[n]{\left(\frac{a}{b^n} + 1\right)^{n-1}} + \sqrt[n]{\left(\frac{a}{b^n} + 1\right)^{n-2} + \dots + 1}} \right) \frac{na^n b^n}{a + b} = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{b}{na^n} + \frac{a}{nb^n} \right) \frac{na^n b^n}{a + b} = \lim_{n \rightarrow \infty} \frac{a^{n+1} + b^{n+1}}{a + b} = \infty
 \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a^n + b} + \sqrt[n]{a + b^n}}{a + b} \right)^{na^n b^n} = e^\infty = \infty$$

Solution 2 by Florentin Vișescu-Romania

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sqrt[n]{a^n + b} &= \lim_{n \rightarrow \infty} (a^n + b)^{\frac{1}{n}} = e^{\lim_{n \rightarrow \infty} \frac{\log(a^n + b)}{n}} = e^{\lim_{n \rightarrow \infty} \left(\log a + \frac{\log(1 + \frac{b}{a^n})}{n} \right)} = e^{\log a} = a \\
 \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a^n + b} + \sqrt[n]{a + b^n}}{a + b} \right)^{na^n b^n} &= \lim_{n \rightarrow \infty} \left(1 + \frac{\sqrt[n]{a^n + b} - a + \sqrt[n]{a + b^n} - b}{a + b} \right)^{na^n b^n} = \\
 &= e^{\frac{1}{a+b} \lim_{n \rightarrow \infty} (\sqrt[n]{a^n + b} - a + \sqrt[n]{a + b^n} - b) na^n b^n}
 \end{aligned}$$



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$$\begin{aligned}
 & \lim_{n \rightarrow \infty} (\sqrt[n]{a^n + b} - a + \sqrt[n]{a + b^n} - b) n a^n b^n = \\
 &= \lim_{n \rightarrow \infty} (e^{\log \sqrt[n]{a^n + b}} - e^{\log a} + e^{\log \sqrt[n]{a + b^n}} - e^{\log b}) n a^n b^n = \\
 &= \lim_{n \rightarrow \infty} \left(e^{\frac{\log(a^n + b)}{n}} - e^{\log a} + e^{\frac{\log(a + b^n)}{n}} - e^{\log b} \right) n a^n b^n = \\
 &= \lim_{n \rightarrow \infty} n a^n b^n \left[\left(e^{\frac{n \log a + \log(1 + \frac{b}{a^n})}{n}} - e^{\log a} \right) + \left(e^{\frac{n \log b + \log(1 + \frac{a}{b^n})}{n}} - e^{\log b} \right) \right] = \\
 &= \lim_{n \rightarrow \infty} n a^n b^n \left[e^{\log a} \left(e^{\frac{\log(1 + \frac{b}{a^n})}{n}} - 1 \right) + e^{\log b} \left(e^{\frac{\log(1 + \frac{a}{b^n})}{n}} - 1 \right) \right] = \\
 &= \lim_{n \rightarrow \infty} n a^n b^n \left[a \left(e^{\frac{\log(1 + \frac{b}{a^n})}{n}} - 1 \right) + b \left(e^{\frac{\log(1 + \frac{a}{b^n})}{n}} - 1 \right) \right] = \\
 &= \lim_{n \rightarrow \infty} n a^n b^n \left[a \left(\underbrace{\frac{e^{\frac{\log(1 + \frac{b}{a^n})}{n}} - 1}{\log(1 + \frac{b}{a^n})}}_{\rightarrow 1} \cdot \frac{\log(1 + \frac{b}{a^n})}{n} \right) + b \left(\underbrace{\frac{e^{\frac{\log(1 + \frac{a}{b^n})}{n}} - 1}{\log(1 + \frac{a}{b^n})}}_{\rightarrow 1} \cdot \frac{\log(1 + \frac{a}{b^n})}{n} \right) \right] \\
 &= \lim_{n \rightarrow \infty} a^n b^n \left[a \cdot \log \left(1 + \frac{b}{a^n} \right) + b \cdot \log \left(1 + \frac{a}{b^n} \right) \right] = \\
 &= \lim_{n \rightarrow \infty} ab^{n-1} \cdot \underbrace{\log \left(1 + \frac{b}{a^n} \right)^{\frac{a^n}{b}}}_{\rightarrow 1} + a^{n-1} b \cdot \underbrace{\log \left(1 + \frac{a}{b^n} \right)^{\frac{b^n}{a}}}_{\rightarrow 1} = \lim_{n \rightarrow \infty} (ab^{n-1} + a^{n-1} b) = \infty
 \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a^n + b} + \sqrt[n]{a + b^n}}{a + b} \right)^{na^n b^n} = e^\infty = \infty$$

1181. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{k=0}^n \sum_{i=0}^k \sum_{j=0}^i (-1)^j \cdot \binom{i}{j} \cdot \frac{3^{i-j}}{4^i} \right)$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$\sum_{j=0}^i (-1)^j \cdot \binom{i}{j} \cdot \frac{3^{i-j}}{4^i} = \sum_{j=0}^i (-1)^j \cdot \binom{i}{j} \cdot \left(\frac{3}{4}\right)^i \cdot \frac{1}{3^j} = \left(\frac{3}{4}\right)^i \cdot \sum_{j=0}^i (-1)^j \cdot \binom{i}{j} \cdot \left(\frac{1}{3}\right)^j =$$

$$= \left(\frac{3}{4}\right)^i \cdot \left(1 - \frac{1}{3}\right)^i = \left(\frac{3}{4}\right)^i \cdot \left(\frac{2}{3}\right)^i = \left(\frac{1}{2}\right)^i$$

$$\sum_{i=0}^k \sum_{j=0}^i (-1)^j \cdot \binom{i}{j} \cdot \frac{3^{i-j}}{4^i} = \sum_{i=0}^k \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^k = 2 \left[1 - \left(\frac{1}{2}\right)^k\right]$$

$$\sum_{k=0}^n \sum_{i=0}^k \sum_{j=0}^i (-1)^j \cdot \binom{i}{j} \cdot \frac{3^{i-j}}{4^i} = 2 \sum_{k=0}^n \left[1 - \left(\frac{1}{2}\right)^k\right] = 2n - 4 \left[1 - \left(\frac{1}{2}\right)^n\right]$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{k=0}^n \sum_{i=0}^k \sum_{j=0}^i (-1)^j \cdot \binom{i}{j} \cdot \frac{3^{i-j}}{4^i} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(2n - 4 \left[1 - \left(\frac{1}{2}\right)^n\right] \right) =$$

$$= \lim_{n \rightarrow \infty} \left(2 - \frac{4}{n} \right) = 2$$

1182. Find:

$$\Omega = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\sum_{k=1}^n k^3 \binom{n}{k}^2 \right) \left(\sum_{k=1}^n k^2 \binom{n}{k}^2 \right)^{-1}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Abdallah El Farissi-Bechar-Algerie

We have for $k \in \{1, 2, \dots, n\}$, $k^2 \binom{n}{k}^2 \leq k^3 \binom{n}{k}^2 \leq nk^2 \binom{n}{k}^2$ then

$$\sum_{k=1}^n k^2 \binom{n}{k}^2 \leq \sum_{k=1}^n k^3 \binom{n}{k}^2 \leq \sum_{k=1}^n nk^2 \binom{n}{k}^2 \Leftrightarrow$$

$$1 \leq \sqrt[n]{\left(\sum_{k=1}^n k^3 \binom{n}{k}^2 \right) \left(\sum_{k=1}^n k^2 \binom{n}{k}^2 \right)^{-1}} \leq n^{\frac{1}{n}}$$



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$$\Omega = \lim_{n \rightarrow \infty} \sqrt{n \left(\sum_{k=1}^n k^3 \binom{n}{k}^2 \right) \left(\sum_{k=1}^n k^2 \binom{n}{k}^2 \right)^{-1}} = 1$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

We'll use the identity:

$$\begin{aligned} \binom{n}{k} &= \int_{|z|=1} \frac{(1+z)^n}{z^{k+1}} \cdot \frac{dz}{2\pi i} \\ \varphi_1(n) &= \sum_{k=1}^n k^2 \binom{n}{k}^2 = \sum_{k=1}^n k^2 \binom{n}{k} \binom{n}{n-k} = \int_{|z|=1} \frac{(1+z)^n}{z^{n+1}} \sum_{k=1}^n k^2 \binom{n}{k} z^k \cdot \frac{dz}{2\pi i} = \\ &= \int_{|z|=1} \frac{(1+z)^n}{z^{n+1}} (nz(1+z)^{n-2}(1+nz)) \frac{dz}{2\pi i} = \\ &= n^2 \int_{|z|=1} \frac{(1+z)^{2n-2}}{z^{n-1}} \cdot \frac{dz}{2\pi i} + n \int_{|z|=1} \frac{(1+z)^{2n-2}}{z^n} \cdot \frac{dz}{2\pi i} = \\ &= n^2 \binom{2n-2}{n-2} + n \binom{2n-2}{n-1} = \frac{4^{n-1} n^2 \Gamma(n - \frac{1}{2})}{\sqrt{\pi} (n-1)!} \end{aligned}$$

Similarly:

$$\begin{aligned} \varphi_2(n) &= \sum_{k=1}^n k^3 \binom{n}{k}^2 = \frac{2^{2n-3} n^2 \Gamma(n - \frac{1}{2})}{\sqrt{\pi} (n-1)!} \\ \Omega &= \lim_{n \rightarrow \infty} \sqrt{n \left(\sum_{k=1}^n k^3 \binom{n}{k}^2 \right) \left(\sum_{k=1}^n k^2 \binom{n}{k}^2 \right)^{-1}} = \lim_{n \rightarrow \infty} \sqrt{\frac{\varphi_2(n)}{\varphi_1(n)}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{2}} = 1 \end{aligned}$$

Solution 3 by Adrian Popa-Romania

$$\begin{aligned} S_1 &= \sum_{k=1}^n k^2 \binom{n}{k}^2 \\ \frac{k}{n} \binom{n}{k} &= \binom{n-1}{k-1} \Rightarrow k \binom{n}{k} = n \binom{n-1}{k-1} \Rightarrow k^2 \binom{n}{k}^2 = n^2 \binom{n-1}{k-1}^2 \Rightarrow \end{aligned}$$



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$$S_1 = \sum_{k=1}^n k^2 \binom{n}{k}^2 = \sum_{k=1}^n n^2 \binom{n-1}{k-1}^2 = n^2 \binom{2n-2}{n-1} = \frac{n^2(2n-2)!}{((n-1)!)^2}$$

$$S_2 = \sum_{k=1}^n k^3 \binom{n}{k}^2 = \sum_{k=1}^n k \cdot k^2 \binom{n}{k}^2 = \sum_{k=1}^n k \cdot n^2 \binom{n-1}{k-1}^2 = n^2 \sum_{k=1}^n k \binom{n-1}{k-1}^2$$

$$\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1} \Rightarrow \frac{k-1}{n-1} \binom{n-1}{k-1} = \binom{n-2}{k-2} \Rightarrow (k-1) \binom{n-1}{k-1} = (n-1) \binom{n-2}{k-2}$$

$$\begin{aligned} S_2 &= n^2 \sum_{k=1}^n k \binom{n-1}{k-1}^2 = n^2 \sum_{k=1}^n (k-1+1) \binom{n-1}{k-1}^2 = \\ &= n^2 \left[\sum_{k=1}^n (k-1) \binom{n-1}{k-1} \binom{n-1}{k-1} + \sum_{k=1}^n \binom{n-1}{k-1}^2 = \right. \\ &= n^2 \left[\sum_{k=2}^n (n-1) \binom{n-2}{k-2} \binom{n-1}{k-1} + \binom{2n-2}{n-1} \right] = \\ &= n^2(n-1) \sum_{k=2}^n \binom{n-2}{k-2} \binom{n-1}{k-1} + n^2 \binom{2n-2}{n-1} \end{aligned}$$

Let's find that:

$$S_3 = \sum_{k=2}^n \binom{n-2}{k-2} \binom{n-1}{k-1}$$

We have that:

$$(1+x)^{n-2} = \sum_{k=0}^n \binom{n-2}{n-k-2} x^k \text{ and } (1+x)^{n-1} = \sum_{k=0}^n \binom{n-1}{n-k-1} x^k$$

$\sum_{k=2}^n \binom{n-2}{k-2} \binom{n-1}{k-1}$ is the coefficient by x^{n-2} from product $(1+x)^{n-2} \cdot (1+x)^{n-1}$

On the other hand, coefficient by x^{n-2} from product

$$(1+x)^{n-2} \cdot (1+x)^{n-1} = (1+x)^{2n-3} \text{ is } \binom{2n-3}{n-2}$$

Therefore,

$$S_3 = \sum_{k=2}^n \binom{n-2}{k-2} \binom{n-1}{k-1} = \binom{2n-3}{n-2}$$

$$\begin{aligned} \frac{\sum_{k=1}^n k^3 \binom{n}{k}^2}{\sum_{k=1}^n k^2 \binom{n}{k}^2} &= \frac{n^2(n-1) \binom{2n-3}{n-2} + n^2 \binom{2n-2}{n-1}}{n^2 \binom{n-2}{n-1}} = (n-1) \frac{\binom{2n-3}{n-2}}{\binom{2n-2}{n-1}} + 1 = \\ &= \frac{(n-1)^2}{2n-2} + 1 = \frac{n+1}{2} \\ &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\sum_{k=1}^n k^3 \binom{n}{k}^2 \right) \left(\sum_{k=1}^n k^2 \binom{n}{k}^2 \right)^{-1}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n+1}{2}} \stackrel{c-d'A}{=} \lim_{n \rightarrow \infty} \frac{n+2}{2} \cdot \frac{2}{n+1} = 1 \end{aligned}$$

1183.

$$x_n = \sum_{k=1}^{n-1} \frac{1}{1+\omega^k}, y_n = \sum_{k=1}^{n-1} \frac{\omega^{2k}}{1+\omega^k}, \omega^n = 1, \omega \neq 1, n \in \mathbb{N}, n \geq 3$$

Find:

$$\Omega = \lim_{n \rightarrow \infty} \frac{e^{H_n}}{x_n - y_n}$$

Proposed by Surjeet Singhania-India

Solution by Adrian Popa-Romania

$$\begin{aligned} x_n - y_n &= \sum_{k=1}^{n-1} \frac{1}{1+\omega^k} - \sum_{k=1}^{n-1} \frac{\omega^{2k}}{1+\omega^k} = \sum_{k=1}^{n-1} \frac{1-\omega^{2k}}{1+\omega^k} = \sum_{k=1}^{n-1} (1-\omega^k) = n-1 - \sum_{k=1}^{n-1} \omega^k = \\ &= n-1 + \omega \cdot \frac{\omega^{n-1}-1}{\omega-1} = n-1 - \frac{\omega^n - \omega}{\omega-1} = n-1 - \frac{1-\omega}{\omega-1} = n \end{aligned}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \frac{e^{H_n}}{x_n - y_n} = \lim_{n \rightarrow \infty} \frac{e^{H_n}}{n} = \lim_{n \rightarrow \infty} \frac{e^{\log n} \cdot e^\gamma}{n} = e^\gamma$$

1184. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left((1-i) \prod_{k=1}^n \frac{k^2 + k + 1 + i}{\sqrt{(k^2 + 1)(k^2 + 2k + 2)}} \right)$$

Proposed by Daniel Sitaru-Romania



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Solution by Adrian Popa-Romania

$$\begin{aligned}
 \prod_{k=1}^n \frac{k^2 + k + 1 + i}{\sqrt{(k^2 + 1)(k^2 + 2k + 2)}} &= \prod_{k=1}^n \frac{(k+i)(k+1-i)}{\sqrt{(k+i)(k-i)(k+1+i)(k+1-i)}} = \\
 &= \prod_{k=1}^n \sqrt{\frac{(k+i)(k+1-i)}{(k-i)(k+1+i)}} = \sqrt{\frac{(1+i)(n+1-i)}{(1-i)(n+1+i)}} \\
 \Omega &= \lim_{n \rightarrow \infty} \left((1-i) \prod_{k=1}^n \frac{k^2 + k + 1 + i}{\sqrt{(k^2 + 1)(k^2 + 2k + 2)}} \right) = \\
 &= \lim_{n \rightarrow \infty} \sqrt{\frac{(1+i)(n+1-i)}{(1-i)(n+1+i)}} = \lim_{n \rightarrow \infty} \sqrt{\frac{2(n+1-i)}{n+1+i}} = \sqrt{2}
 \end{aligned}$$

1185. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{8^n}{n(2n+1)^2} \cdot \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) \cdot \prod_{k=1}^n \sin^2\left(\frac{k\pi}{2n+1}\right) \cdot \prod_{k=n+1}^{2n} \sin\left(\frac{k\pi}{2n+1}\right) \right)$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$P = \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \sin\frac{\pi}{n} \cdot \sin\frac{2\pi}{n} \cdot \dots \cdot \sin\frac{(n-1)\pi}{n}$$

Let be the equation: $x^n - 1 = 0 \Leftrightarrow x^n = \cos 0 + i \sin 0 \Rightarrow$

$$\begin{aligned}
 x_k &= \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k \in \{0, 1, \dots, (n-1)\} \\
 x^n - 1 &= (x-1)(x-x_1)(x-x_2) \cdot \dots \cdot (x-x_n) = \\
 &= (x-1) \left(x - \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n} \right) \cdot \dots \cdot \left(x - \cos \frac{2(n-1)\pi}{n} - i \sin \frac{2(n-1)\pi}{n} \right); \quad (1)
 \end{aligned}$$

$$\text{But } \frac{x^n - 1}{x-1} = x^{n-1} + x^{n-2} + \dots + x + 1; \quad (2)$$

From (1), (2) and $x = 1$, we get:

$$n = \prod_{k=1}^{n-1} \left(1 - \cos \frac{2k\pi}{n} - i \sin \frac{2k\pi}{n} \right) = \prod_{k=1}^{n-1} \left(2 \sin^2 \frac{k\pi}{n} - 2i \sin \frac{k\pi}{n} \cos \frac{k\pi}{n} \right) =$$



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$$\begin{aligned}
 &= \prod_{k=1}^{n-1} 2 \sin \frac{k\pi}{n} \left(\sin \frac{k\pi}{n} - i \cos \frac{k\pi}{n} \right) = \\
 &= \frac{2^{n-1}}{i^{n-1}} \cdot \sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \cdot \dots \cdot \sin \frac{(n-1)\pi}{n} \cdot \prod_{k=1}^{n-1} \left(\cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \right)^k = \\
 &= \frac{2^{n-1}}{i^{n-1}} \cdot P \cdot \left(\cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \right)^{1+2+\dots+n-1} = \\
 &= \frac{2^{n-1}}{i^{n-1}} \cdot P \cdot \left(\cos \frac{n(n-1)\pi}{2n} + i \sin \frac{n(n-1)\pi}{2n} \right) = \\
 &= \frac{\cos 0 + i \sin 0}{\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{n-1}} \cdot 2^{n-1} \cdot P \cdot \left(\cos \frac{(n-1)\pi}{2n} + i \sin \frac{(n-1)\pi}{2n} \right) = \\
 &= \left(\cos \frac{(n-1)\pi}{2n} - i \sin \frac{(n-1)\pi}{2n} \right) \cdot 2^{n-1} \cdot P \cdot \left(\cos \frac{(n-1)\pi}{2n} + i \sin \frac{(n-1)\pi}{2n} \right) = \\
 &= 2^{n-1} \cdot P . \text{ Hence,}
 \end{aligned}$$

$$P = \prod_{k=1}^{n-1} \sin \left(\frac{k\pi}{n} \right) = \sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \cdot \dots \cdot \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$

Therefore,

$$\prod_{k=1}^n \sin \left(\frac{k\pi}{n+1} \right) = \frac{n+1}{2^n} \Rightarrow \prod_{k=1}^n \sin^2 \frac{k\pi}{n+1} = \left(\prod_{k=1}^n \sin \left(\frac{k\pi}{n+1} \right) \right)^2 = \frac{(n+1)^2}{2^{2n}}$$

$$\prod_{k=n+1}^{2n} \sin \left(\frac{k\pi}{2n+1} \right) = \frac{\prod_{k=1}^{2n} \sin \left(\frac{k\pi}{2n+1} \right)}{\prod_{k=1}^n \sin \left(\frac{k\pi}{2n+1} \right)} = \frac{\frac{2n+1}{2^{2n}}}{P_2} = \frac{\frac{2n+1}{2^{2n}}}{\frac{\sqrt{2n+1}}{2^n}} = \frac{\sqrt{2n+1}}{2^n}$$

$$P_2 = \prod_{k=1}^n \sin \left(\frac{k\pi}{2n+1} \right) = \prod_{k=1}^n \sin \left(\pi - \frac{k\pi}{2n+1} \right) = \frac{\sqrt{2n+1}}{2^n}$$

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \left(\frac{8^n}{n(2n+1)^2} \cdot \prod_{k=1}^{n-1} \sin \left(\frac{k\pi}{n} \right) \cdot \prod_{k=1}^n \sin^2 \left(\frac{k\pi}{2n+1} \right) \cdot \prod_{k=n+1}^{2n} \sin \left(\frac{k\pi}{2n+1} \right) \right) = \\
 &= \lim_{n \rightarrow \infty} \frac{8^n}{n(2n+1)^2} \cdot \frac{2n}{2^n} \cdot \frac{(n+1)^2}{2^{2n}} \cdot \frac{\sqrt{2n+1}}{2^n} =
 \end{aligned}$$



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$$= \lim_{n \rightarrow \infty} \frac{2(n+1)^2}{(2n+1)^2} \cdot \frac{\sqrt{2n+1}}{2^n} = 0$$

1186. Find:

$$\Omega = \lim_{n \rightarrow \infty} \frac{\sqrt{(1+n!)^n!}}{n \cdot (n!)!}$$

Proposed by Daniel Sitaru-Romania

Solution by Ahmed Yackoube Chach-Lille-France

$$\begin{aligned} \Omega_n &= \frac{\sqrt{(1+n!)^n!}}{n \cdot (n!)!} = \frac{\sqrt{(n!)^n!} \sqrt{\left(1 + \frac{1}{n!}\right)^{n!}}}{n \sqrt{2\pi n!} \left(\frac{n!}{e}\right)^{n!}} = \frac{\sqrt{\left(1 + \frac{1}{n!}\right)^{n!}} \sqrt{n!^{n!}} e^{n!}}{n \sqrt{2\pi} \sqrt{n!^{n!+1}}} = \\ &= \frac{\sqrt{\left(1 + \frac{1}{n!}\right)^{n!}}}{n \sqrt{2\pi}} e^{n! - \frac{n!+1}{2} \log(n!)} \\ &\lim_{n \rightarrow \infty} \sqrt{\left(1 + \frac{1}{n!}\right)^{n!}} = \sqrt{e} \\ &\lim_{n \rightarrow \infty} e^{n! - \frac{n!+1}{2} \log(n!)} = e^{-\infty} \end{aligned}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \frac{\sqrt{(1+n!)^n!}}{n \cdot (n!)!} = \lim_{n \rightarrow \infty} \frac{1}{n} \sqrt{\frac{e}{2\pi}} e^{-\infty} = 0$$

1187. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be positive real sequences such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{n} = a \in \mathbb{R}_+^* \text{ and } \lim_{n \rightarrow \infty} \frac{b_{n+1}}{nb_n} = b \in \mathbb{R}_+^*$$

Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}} - \frac{a_n}{\sqrt[n]{b_n}} \right)$$

Proposed by D.M.Bătinețu-Giurgiu and Neculae Stanciu-Romania



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Solution 1 by Marian Ursărescu-Romania

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}} - \frac{a_n}{\sqrt[n]{b_n}} \right) = \Omega = \lim_{n \rightarrow \infty} \frac{a_n}{\sqrt[n]{b_n}} \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} - 1 \right) = \\ = \lim_{n \rightarrow \infty} \frac{a_n}{n^2} \cdot \frac{n}{\sqrt[n]{b_n}} \cdot n \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} - 1 \right); \quad (1)$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{n^2} \stackrel{c-s}{=} \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{(n+1)^2 - n^2} = \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{n} \cdot \frac{n}{2n+1} = \frac{a}{2}; \quad (2)$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{b_n}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n^n}{b_n}} \stackrel{c-d}{=} \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{b_{n+1}} \cdot \frac{b_n}{n^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n \cdot \frac{n+1}{n} \cdot \frac{nb_n}{b_{n+1}} = \frac{e}{b}; \quad (3)$$

$$\lim_{n \rightarrow \infty} n \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} - 1 \right) = \lim_{n \rightarrow \infty} n \left(e^{\log \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} \right)} - 1 \right) = \\ = \lim_{n \rightarrow \infty} n \left(\frac{e^{\log \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} \right)} - 1}{\log \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} \right)} \right) \log \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} \right) = \\ = \lim_{n \rightarrow \infty} n \log \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} \right) = \lim_{n \rightarrow \infty} \log \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} \right)^n = \\ = \log \left(\lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)^n \cdot \frac{b_n}{b_{n+1}} \cdot \sqrt[n+1]{b_{n+1}} \right) = \log \left(\lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)^n \cdot \frac{nb_n}{b_{n+1}} \cdot \frac{\sqrt[n+1]{b_{n+1}}}{n} \right); \quad (4)$$

$$\lim_{n \rightarrow \infty} \frac{nb_n}{b_{n+1}} = \frac{1}{b}; \quad (5)$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{b_{n+1}}}{n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{\sqrt[n+1]{b_{n+1}}}{n+1} \stackrel{(3)}{=} \frac{b}{e}; \quad (6)$$

$$\lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{a_{n+1} - a_n}{a_n} \right)^n = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{a_{n+1} - a_n}{a_n} \right)^{\frac{a_n}{a_{n+1} - a_n}} \right]^{n \cdot \frac{a_{n+1} - a_n}{a_n}} = \\ = e^{\lim_{n \rightarrow \infty} \frac{n^2}{a_n} \cdot \frac{a_{n+1} - a_n}{n}} = e^2; \quad (7)$$

From (4) + (5) + (6) + (7) we get:



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$$\lim_{n \rightarrow \infty} n \log \left(\frac{a_{n+1}}{a_n} \cdot \frac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}} - 1 \right) = \log \left(e^2 \cdot \frac{1}{b} \cdot \frac{b}{e} \right) = \log e = 1; \quad (8)$$

From (1) + (2) + (3) + (8) we get: $\Omega = \frac{a}{2} \cdot \frac{e}{b} \cdot 1 = \frac{ae}{2b}$

Solution 2 by Nassim Taleb-New York-USA

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}} - \frac{a_n}{\sqrt[n]{b_n}} \right)$$

We replace $a_{n+1} = a \cdot n + a_n$, $a_n = -\frac{na}{2} + \frac{n^2 a}{2} + c_1$ (from the recurrence equation).

$b_n = b^{n-1} c_2 (n-1)!$ where c_1, c_2 – are constants and further write $(n-1)!$ as $\Gamma(n)$.

Allora: since $\lim_{n \rightarrow \infty} n \Gamma(n)^{-\frac{1}{n+1}} = e$ (via the Stirling approximation before taking the limit),

$$\text{likewise } \lim_{n \rightarrow \infty} n^{-\frac{1}{n+1}} \Gamma(n)^{-\frac{1}{n+1}} = 0, \text{ etc.,}$$

We end up (unless/made a mistake adding something somewhere) with

$$\Omega = \frac{1}{2} \cdot \frac{a}{b} e$$

1188. Let be $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}, a_n, b_n \in \mathbb{R}_+^* = (0, \infty)$ such that

$\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R}_+^*$ and $(b_n)_{n \geq 1}$ is a bounded sequence. If

$(x_n)_{n \geq 1}, x_n = \prod_{k=1}^n (ka_k + b_k)$ find:

$$\lim_{n \rightarrow \infty} (\sqrt[n+1]{x_{n+1}} - \sqrt[n]{x_n})$$

Proposed by D.M. Bătinețu-Giurgiu and Daniel Sitaru-Romania

Solution by Marian Ursărescu-Romania

$$L = \lim_{n \rightarrow \infty} (\sqrt[n+1]{x_{n+1}} - \sqrt[n]{x_n}) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{x_n}}{n} \cdot n \left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} - 1 \right); \quad (1)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt[n]{x_n}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{x_n}{n^n}} \stackrel{C-D}{=} \lim_{n \rightarrow \infty} \frac{x_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x_n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \cdot \frac{x_{n+1}}{(n+1)x_n} = \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \cdot \frac{(n+1)a_{n+1} + b_{n+1}}{n+1} = \frac{a}{e}; \quad (2) \end{aligned}$$



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Because $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \frac{1}{e}$ and $\lim_{n \rightarrow \infty} \frac{(n+1)a_{n+1} + b_{n+1}}{n+1} = \lim_{n \rightarrow \infty} \left(a_{n+1} + \frac{b_{n+1}}{n+1} \right) = a, b_n -\text{is bounded.}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} n \left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} - 1 \right) &= \lim_{n \rightarrow \infty} n \left(e^{\log \left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} \right)} - 1 \right) = \\
 &= \lim_{n \rightarrow \infty} \left(\frac{e^{\log \left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} \right)} - 1}{\log \left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} \right)} \right) n \log \left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} \right) = \\
 &= \lim_{n \rightarrow \infty} \log \left(\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} \right)^n = \log \left(\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} \cdot \frac{1}{\sqrt[n+1]{x_{n+1}}} \right) = \\
 &= \log \left(\lim_{n \rightarrow \infty} \frac{(n+1)a_{n+1} + b_{n+1}}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{b_{n+1}}} \right) = \\
 &= \log \left(a \cdot \frac{e}{a} \right) = \log e = 1; (3)
 \end{aligned}$$

From (1), (2), (3) we get: $L = \frac{a}{e}$

1189. Find a closed form:

$$\Omega = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^4 n^2 (m^2 + n^2)}$$

Proposed by Surjeet Singhania-India

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned}
 \Omega &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^4 n^2 (m^2 + n^2)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^4 m^2 (m^2 + n^2)} \\
 2\Omega &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m^2 + n^2)} \left(\frac{1}{m^4 n^2} + \frac{1}{m^2 n^4} \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 + n^2} \cdot \frac{m^2 + n^2}{m^4 n^4} = \\
 &= \left(\sum_{m=1}^{\infty} \frac{1}{m^4} \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^4} \right) = \frac{\pi^8}{8100}
 \end{aligned}$$



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$$\left(\because \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \right)$$

$$\Omega = \frac{\pi^8}{16200}$$

Solution 2 by Precious Itsuokor-Nigeria

$$\begin{aligned} \Omega &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^4 n^2 (m^2 + n^2)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^4 m^2 (m^2 + n^2)} \\ 2\Omega &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m^2 + n^2)} \left(\frac{1}{m^4 n^2} + \frac{1}{m^2 n^4} \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 + n^2} \cdot \frac{m^2 + n^2}{m^4 n^4} = \\ &= \left(\sum_{m=1}^{\infty} \frac{1}{m^4} \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^4} \right) = \zeta(4) \zeta(4) = \frac{\pi^8}{8100} \\ \Omega &= \frac{\pi^8}{16200} \end{aligned}$$

1190. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sergio Esteban-Argentina

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\frac{1}{(2n+1)!!}} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\frac{2^n n!}{(2n+1)!}} \right) = \lim_{n \rightarrow \infty} \left(\frac{2}{\log(n+1)} \sqrt[n]{\frac{n!}{(2n+1)!}} \right) \text{ by Stirling's} = \\ &= \lim_{n \rightarrow \infty} \left(\frac{2}{\log(n+1)} \cdot \frac{\frac{n}{e}}{\left(\frac{2n+1}{e}\right)^{\frac{2n+1}{n}}} \right) = \frac{2}{e} \cdot \lim_{n \rightarrow \infty} \frac{n}{\log(n+1)} \cdot \frac{e^2}{(2n+1)^2} = 0 \end{aligned}$$



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Solution 2 by Asmat Quatea-Kabul-Afghanistan

$$\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot 2n}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} = \frac{(2n)!}{2^n \cdot n!}$$

$$\frac{1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} = \frac{2^n \cdot n!}{(2n)!}$$

$$\frac{1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} = \frac{2^n \cdot n!}{(2n)! (2n+1)}$$

$$n! \approx \sqrt{2n\pi} \left(\frac{n}{e}\right)^n ; (2n)! = \sqrt{4n\pi} \left(\frac{2n}{e}\right)^{2n}$$

$$\frac{1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} \approx \frac{2^n \cdot \sqrt{2n\pi} \left(\frac{n}{e}\right)^n}{(2n+1)\sqrt{4n\pi} \left(\frac{2n}{e}\right)^{2n}}$$

$$\frac{1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} \approx \frac{2^n \cdot \sqrt{2} \left(\frac{n}{e}\right)^n}{(4n+2) \left(\frac{2n}{e}\right)^{2n}}$$

$$\left(\frac{1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} \right)^n \approx \frac{2 \cdot (\sqrt{2})^{\frac{1}{n}} \cdot \frac{n}{e}}{(4n+2)^{\frac{1}{n}} \left(\frac{4n^2}{e^2}\right)}$$

$$\left(\frac{1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} \right)^n \approx \frac{2 \cdot \frac{n}{e}}{\frac{4n^2}{e^2}} = \frac{e}{2n}$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \cdot \frac{e}{2n} \right) = 0$$

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \right) = 0$$

Solution 3 by Florentin Vișescu-Romania

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \right) = \lim_{n \rightarrow \infty} \left(\sqrt[n]{\frac{1}{\log^n(n+1)} \cdot \prod_{k=1}^n \frac{1}{2k+1}} \right)$$



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$$a_n = \frac{1}{\log^n(n+1)} \cdot \prod_{k=1}^n \frac{1}{2k+1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{1}{\log^{n+1}(n+2)} \cdot \prod_{k=1}^{n+1} \frac{1}{2k+1} \cdot \frac{\log^n(n+1)}{1} \cdot \frac{1}{\prod_{k=1}^n \frac{1}{2k+1}} = \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2n+3)\log(n+2)} \cdot \left(\frac{\log(n+1)}{\log(n+2)} \right)^n = 0 \\ \lim_{n \rightarrow \infty} \left(\frac{\log(n+1)}{\log(n+2)} \right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{\log \left(\frac{n+1}{n+2} \right)}{\log(n+2)} \right)^n = e^{\lim_{n \rightarrow \infty} \left(\frac{\log \left(\frac{n+1}{n+2} \right)^n}{\log(n+2)} \right)} = \\ &= e^{\lim_{n \rightarrow \infty} \left(\frac{\log \left[\left(1 + \frac{-1}{n+2} \right)^{n+2} \right]^{\frac{n}{n+2}}}{\log(n+2)} \right)} = e^0 = 1 \end{aligned}$$

Solution 4 by Ravi Prakash-New Delhi-India

$$a_n = \prod_{k=1}^n \frac{1}{2k+1}$$

$$\text{For } 1 \leq k \leq n \Rightarrow \frac{1}{2n+1} \leq \frac{1}{2k+1} \leq \frac{1}{3}$$

$$\frac{1}{(2n+1)^n} \leq \frac{1}{(2k+1)^n} \leq \frac{1}{3^n}$$

$$\frac{1}{2n+1} \leq \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \leq \frac{1}{3}$$

$$\frac{1}{(2n+1)\log(n+1)} \leq \frac{1}{\log(2n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \leq \frac{1}{(2n+1)\log(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{1}{(2n+1)\log(n+1)} = \lim_{n \rightarrow \infty} \frac{1}{(2n+1)\log(n+1)} = 0$$

Therefore,



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$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \right) = 0$$

Solution 5 by Abdallah El Farissi-Bechar-Algerie

$$\frac{1}{n+2} = \frac{n}{\sum_{k=1}^n (2k+1)} \leq \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \leq \frac{\sum_{k=1}^n \frac{1}{2k+1}}{n} \leq \frac{1}{3}$$

$$\frac{1}{(n+2)\log(n+1)} \leq \frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \leq \frac{1}{3\log(n+1)}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \right) = 0$$

Solution 6 by Remus Florin Stanca-Romania

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \right) = \lim_{n \rightarrow \infty} \left(\sqrt[n]{\frac{1}{\log^n(n+1)} \cdot \prod_{k=1}^n \frac{1}{2k+1}} \right) \stackrel{C-D}{=} \\ &= \lim_{n \rightarrow \infty} \frac{\prod_{k=1}^{n+1} \frac{1}{2k+1}}{\log^{n+1}(n+2)} \cdot \frac{\log^n(n+1)}{\prod_{k=1}^n \frac{1}{2k+1}} = \lim_{n \rightarrow \infty} \frac{1}{(2n+3)\log(n+2)} \cdot \left(\frac{\log(n+1)}{\log(n+2)} \right)^n = \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2n+3)\log(n+2)} \cdot \left(1 + \frac{\log(n+1) - \log(n+2)}{\log(n+2)} \right)^n = \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2n+3)\log(n+2)} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{\log(n+1) - \log(n+2)}{\log(n+2)} \right)^n = \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2n+3)\log(n+2)} \cdot \lim_{n \rightarrow \infty} e^{\frac{n}{\log(n+2)} \log(\frac{n+1}{n+2})} = \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2n+3)\log(n+2)} \cdot \lim_{n \rightarrow \infty} e^{\frac{-1}{\log(n+2)}} = 0 \end{aligned}$$

Therefore,



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$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{\log(n+1)} \sqrt[n]{\prod_{k=1}^n \frac{1}{2k+1}} \right) = 0$$

1191. Find:

$$\Omega = \lim_{n \rightarrow \infty} \frac{12^n \cdot (n!)^5}{(n^2 + n)^{2n} \cdot (n+1)^n}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Asmat Oatea-Afghanistan

$$\begin{aligned}
 n! &\cong \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \\
 \Omega &= \lim_{n \rightarrow \infty} \frac{12^n \cdot (n!)^5}{(n^2 + n)^{2n} \cdot (n+1)^n} = \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{2\pi n})^5 \left(\frac{n}{e}\right)^{5n}}{(n^4 + 2n^3 + n^2)^n \cdot (n+1)^n} = \\
 &= \left(\sqrt{32\pi^5}\right) \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5 n^{5n} \cdot \frac{1}{e^{5n}}}{(n^5 + 3n^4 + 3n^3 + n^2)^n} = \left(\sqrt{32\pi^5}\right) \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5 n^{5n} \cdot \frac{1}{e^{5n}}}{n^{5n} \left(1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}\right)^n} = \\
 &= \left(\sqrt{32\pi^5}\right) \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5 \cdot \frac{1}{e^{5n}}}{\left(1 + \frac{3n^2 + 3n + 1}{n^3}\right)^n} = \\
 &= \left(\sqrt{32\pi^5}\right) \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5 \cdot \frac{1}{e^{5n}}}{\left[\left(1 + \frac{3n^2 + 3n + 1}{n^3}\right)^{\frac{n^3}{3n^2+3n+1}n}\right]^{\frac{3n^2+3n+1}{n^3}}} = \\
 &= \left(\sqrt{32\pi^5}\right) \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5 \cdot \frac{1}{e^{5n}}}{e^{\frac{3n^2+3n+1}{n^2}}} = \left(\sqrt{32\pi^5}\right) \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5 \cdot \frac{1}{e^{5n}}}{e^3} = \\
 &= \left(\frac{\sqrt{32\pi^5}}{e^3}\right) \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5}{e^{5n}} = \left(\frac{\sqrt{32\pi^5}}{e^3}\right) \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5}{e^{4n} \cdot e} = 0,
 \end{aligned}$$

note: $e^{4n} > 12^n$; $e^n > \sqrt{n^5}$



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Solution 2 by Precious Itsuokor-Nigeria

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \frac{12^n \cdot (n!)^5}{(n^2 + n)^{2n} \cdot (n + 1)^n} = \lim_{n \rightarrow \infty} \frac{12^n \cdot (n!)^5}{n^{2n} \cdot (n + 1)^{3n}} = \\
 &= \lim_{n \rightarrow \infty} \frac{12^n \cdot (n!)^5}{n^{5n} \cdot \left(1 + \frac{1}{n}\right)^{3n}} = \lim_{n \rightarrow \infty} \frac{12^n \cdot \left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right)^5}{n^{5n} \cdot \left(1 + \frac{1}{n}\right)^{3n}} = \\
 &= 4\pi^2 \sqrt{2\pi} \lim_{n \rightarrow \infty} (12^n \cdot n^2 \sqrt{n} \cdot e^{-5n}) \left(1 + \frac{1}{n}\right)^{-3n} = \\
 &= 4\pi^2 \sqrt{2\pi} e^{-3} \lim_{n \rightarrow \infty} \frac{n^{\frac{5}{2}}}{\left(\frac{e^5}{12}\right)^n} \stackrel{L'H}{=} \frac{4\pi^2 \sqrt{2\pi} e^{-3} \cdot 15}{8 \log^3\left(\frac{e^5}{12}\right)} \cdot \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} \cdot \log^3\left(\frac{e^5}{12}\right)} = 0
 \end{aligned}$$

Solution 3 by Hussain Reza Zadah-Afghanistan

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \frac{12^n \cdot (n!)^5}{(n^2 + n)^{2n} \cdot (n + 1)^n} = (\sqrt{2\pi})^5 \lim_{n \rightarrow \infty} \frac{12^n \cdot n^{5n} \cdot (\sqrt{n})^5}{n^{2n} \cdot (n + 1)^n \cdot e^{5n}} = \\
 &= (\sqrt{2\pi})^5 \lim_{n \rightarrow \infty} \frac{12^n \cdot n^{3n} \cdot (\sqrt{n})^5}{(n + 1)^{3n} \cdot e^{5n}} = (\sqrt{2\pi})^5 \lim_{n \rightarrow \infty} \frac{12^n \cdot (\sqrt{n})^5}{e^{5n}} \cdot \left(\frac{n}{n + 1}\right)^{3n} = \\
 &= (\sqrt{2\pi})^5 \lim_{n \rightarrow \infty} \left(\frac{12}{e^4}\right)^n \cdot \frac{(\sqrt{n})^5}{e^n} \cdot \left(\frac{n}{n + 1}\right)^{3n} = 0
 \end{aligned}$$

Solution 4 by Ali Jaffal-Lebanon

$$\begin{aligned}
 \text{Let: } u_n &= \frac{12^n \cdot (n!)^5}{(n^2 + n)^{2n} \cdot (n + 1)^n} \\
 \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x &= e^a \\
 (n + 1)^n &= n^n \left(1 + \frac{1}{n}\right)^n \cong n^n e \\
 (n^2 + n)^{2n} &= n^{2n} (1 + n)^{2n} = n^{4n} \left(1 + \frac{1}{n}\right)^{2n} \cong n^{4n} \cdot e^2 \\
 \text{Then } (n^2 + n)^{2n} \cdot (n + 1)^n &\cong e^3 \cdot n^{5n} \\
 n! &\cong \sqrt{2\pi n} \left(\frac{n}{e}\right)^n
 \end{aligned}$$



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So, $(n!)^5 \cong (2\pi n)^{\frac{5}{2}} \cdot \frac{n^{5n}}{e^{5n}} \cong \frac{(2\pi)^{\frac{5}{2}} \cdot n^{\frac{5}{2}} \cdot n^{5n}}{e^{5n}}$. Therefore,

$$\lim_{n \rightarrow \infty} \frac{12^n \cdot (2\pi)^{\frac{5}{2}} \cdot n^{\frac{5}{2}} \cdot n^{5n}}{e^3 \cdot n^{5n} \cdot e^{5n}} = \lim_{n \rightarrow \infty} \frac{12^n \cdot (2\pi)^{\frac{5}{2}} \cdot n^{\frac{5}{2}}}{e^3 \cdot n^{5n}}$$

We know that:

$$\lim_{n \rightarrow \infty} \left(\frac{12}{e^5}\right)^n n^{\frac{5}{2}} = e^{\lim_{n \rightarrow \infty} n \log\left(\frac{12}{e^5}\right) + \frac{5}{2} \log n} = e^{\lim_{n \rightarrow \infty} n \left[\log\left(\frac{12}{e^5}\right) + \frac{\frac{5}{2} \log n}{n}\right]} = e^{-\infty} = 0$$

Then,

$$\Omega = \lim_{n \rightarrow \infty} \frac{12^n \cdot (n!)^5}{(n^2 + n)^{2n} \cdot (n + 1)^n} = 0$$

Solution 5 by Abdallah El Farissi-Algerie

If $\lim_{n \rightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = l$ and $l < 1$, then $\lim_{n \rightarrow \infty} a_n = 0$.

$$a_n = \frac{12^n \cdot (n!)^5}{(n^2 + n)^{2n} \cdot (n + 1)^n} = \frac{12^n \cdot (n!)^5}{n^{2n} \cdot (n + 1)^{3n}}$$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{12^{n+1} \cdot ((n+1)!)^5}{(n+1)^{2(n+1)} \cdot (n+2)^{3n}} \cdot \frac{n^{2n} \cdot (n+1)^{3n}}{12^n \cdot (n!)^5} = \\ &= 12 \left(\frac{n+1}{n+2}\right)^3 \left(\frac{n+1}{n+2}\right)^n \left(\frac{n}{n+2}\right)^{2n} = \\ &= 12 \left(\frac{n+1}{n+2}\right)^3 \left(1 - \frac{1}{n+2}\right)^n \left(1 - \frac{2}{n+2}\right)^{2n} \rightarrow \frac{12}{e^5} < 1. \text{ Therefore,} \end{aligned}$$

$$\Omega = \lim_{n \rightarrow \infty} \frac{12^n \cdot (n!)^5}{(n^2 + n)^{2n} \cdot (n + 1)^n} = 0$$

1192. Find:

$$\Omega = \lim_{x \rightarrow 1} \frac{(\tan \sqrt{3x+2} - \tan \sqrt{2x+3})(\tan \sqrt{4x+3} - \tan \sqrt{3x+4})}{(\tan \sqrt{5x+4} - \tan \sqrt{4x+5})(\tan \sqrt{6x+5} - \tan \sqrt{5x+6})}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Asmat Qatea-Kabul-Afghanistan

$$\Omega = \lim_{x \rightarrow 1} \frac{(\tan \sqrt{3x+2} - \tan \sqrt{2x+3})(\tan \sqrt{4x+3} - \tan \sqrt{3x+4})}{(\tan \sqrt{5x+4} - \tan \sqrt{4x+5})(\tan \sqrt{6x+5} - \tan \sqrt{5x+6})} = \frac{S_1 \cdot S_2}{S_3 \cdot S_4}$$



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We know that:

$$\lim_{a \rightarrow b} \frac{\tan a - \tan b}{a - b} = \sec^2 b$$

$$\begin{aligned} & \lim_{x \rightarrow 1} (\tan \sqrt{dx + c} - \tan \sqrt{cx + d}) = \\ &= \lim_{x \rightarrow 1} \frac{\tan \sqrt{dx + c} - \tan \sqrt{cx + d}}{\sqrt{dx + c} - \sqrt{cx + d}} (\sqrt{dx + c} - \sqrt{cx + d}) = \\ &= \lim_{x \rightarrow 1} (\sqrt{dx + c} - \sqrt{cx + d}) \sec^2 \sqrt{c+d} \end{aligned}$$

$$S_1 = \lim_{x \rightarrow 1} (\tan \sqrt{3x+2} - \tan \sqrt{2x+3}) = \lim_{x \rightarrow 1} (\sqrt{3x+2} - \sqrt{2x+3}) \sec^2 \sqrt{5}$$

$$S_2 = \lim_{x \rightarrow 1} (\tan \sqrt{4x+3} - \tan \sqrt{3x+4}) = \lim_{x \rightarrow 1} (\sqrt{4x+3} - \sqrt{3x+4}) \sec^2 \sqrt{7}$$

$$S_3 = \lim_{x \rightarrow 1} (\tan \sqrt{5x+4} - \tan \sqrt{4x+5}) = \lim_{x \rightarrow 1} (\sqrt{5x+4} - \sqrt{4x+5}) \sec^2 3$$

$$S_4 = \lim_{x \rightarrow 1} (\tan \sqrt{6x+5} - \tan \sqrt{5x+6}) = \lim_{x \rightarrow 1} (\sqrt{6x+5} - \sqrt{5x+6}) \sec^2 \sqrt{11}$$

$$\alpha = \frac{\sec^2 \sqrt{5} \cdot \sec^2 \sqrt{7}}{\sec^2 3 \cdot \sec^2 \sqrt{11}} = \left(\frac{\cos 3 \cdot \cos \sqrt{11}}{\cos \sqrt{5} \cdot \cos \sqrt{7}} \right)^2$$

$$\frac{S_1 \cdot S_2}{S_3 \cdot S_4} = \alpha \cdot \lim_{x \rightarrow 1} \frac{(\sqrt{3x+2} - \sqrt{2x+3})(\sqrt{4x+3} - \sqrt{3x+4})}{(\sqrt{5x+4} - \sqrt{4x+5})(\sqrt{6x+5} - \sqrt{5x+6})}$$

$$\lim_{x \rightarrow 1} \frac{\frac{(\sqrt{3x+2} - \sqrt{2x+3})(\sqrt{3x+2} + \sqrt{2x+3})}{2\sqrt{5}}}{\frac{(\sqrt{5x+4} - \sqrt{4x+5})(\sqrt{5x+4} + \sqrt{4x+5})}{2\sqrt{9}}} = \lim_{x \rightarrow 1} \frac{\frac{x-1}{2\sqrt{5}}}{\frac{x-1}{2\sqrt{9}}} = \frac{3}{\sqrt{5}}$$

$$\lim_{x \rightarrow 1} \frac{\frac{(\sqrt{4x+3} - \sqrt{3x+4})(\sqrt{4x+3} + \sqrt{3x+4})}{2\sqrt{7}}}{\frac{(\sqrt{6x+5} - \sqrt{5x+6})(\sqrt{6x+5} + \sqrt{5x+6})}{2\sqrt{11}}} = \lim_{x \rightarrow 1} \frac{\frac{x-1}{2\sqrt{7}}}{\frac{x-1}{2\sqrt{11}}} = \sqrt{\frac{11}{7}}$$

Therefore,

$$\begin{aligned} & \lim_{x \rightarrow 1} \frac{(\tan \sqrt{3x+2} - \tan \sqrt{2x+3})(\tan \sqrt{4x+3} - \tan \sqrt{3x+4})}{(\tan \sqrt{5x+4} - \tan \sqrt{4x+5})(\tan \sqrt{6x+5} - \tan \sqrt{5x+6})} = \\ &= \frac{3}{\sqrt{5}} \cdot \sqrt{\frac{11}{7}} \cdot \left(\frac{\cos 3 \cdot \cos \sqrt{11}}{\cos \sqrt{5} \cdot \cos \sqrt{7}} \right)^2 \end{aligned}$$



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Solution 2 by Florentin Vișescu-Romania

Firstly, we find:

$$\lim_{x \rightarrow 1} \frac{\tan\sqrt{(n+1)x+n} - \tan\sqrt{nx+(n+1)}}{x-1} = \Phi$$

For $x > 1$ let be the function $f: [nx + (n+1), (n+1)x+n] \rightarrow \mathbb{R}$, $f(t) = \tan\sqrt{t}$

From M.V.T. $\exists c \in (nx + (n+1), (n+1)x+n)$ such that:

$$\frac{\tan\sqrt{(n+1)x+n} - \tan\sqrt{nx+(n+1)}}{x-1} = f'(c) \Leftrightarrow$$

$$\frac{\tan\sqrt{(n+1)x+n} - \tan\sqrt{nx+(n+1)}}{x-1} = \frac{1}{2\sqrt{c}} \cdot \frac{1}{\cos^2\sqrt{c}}$$

For $x = 1$ we have: $\begin{cases} nx + (n+1) = 2n+1 \\ (n+1)x+n = 2n+1 \end{cases} \Rightarrow c \rightarrow 2n+1$

$$\lim_{x \rightarrow 1} \frac{\tan\sqrt{(n+1)x+n} - \tan\sqrt{nx+(n+1)}}{x-1} = \frac{1}{2\sqrt{n+1}} \cdot \frac{1}{\cos^2\sqrt{2n+1}}$$

Therefore,

$$\begin{aligned} \Omega &= \lim_{x \rightarrow 1} \frac{(\tan\sqrt{3x+2} - \tan\sqrt{2x+3})(\tan\sqrt{4x+3} - \tan\sqrt{3x+4})}{(\tan\sqrt{5x+4} - \tan\sqrt{4x+5})(\tan\sqrt{6x+5} - \tan\sqrt{5x+6})} = \\ &= \frac{3}{\sqrt{5}} \cdot \sqrt{\frac{11}{7}} \cdot \left(\frac{\cos 3 \cdot \cos \sqrt{11}}{\cos \sqrt{5} \cdot \cos \sqrt{7}} \right)^2 \end{aligned}$$

Solution 3 by Mohammad Rostami-Afghanistan

$$\begin{aligned} \Omega &= \lim_{x \rightarrow 1} \frac{(\tan\sqrt{3x+2} - \tan\sqrt{2x+3})(\tan\sqrt{4x+3} - \tan\sqrt{3x+4})}{(\tan\sqrt{5x+4} - \tan\sqrt{4x+5})(\tan\sqrt{6x+5} - \tan\sqrt{5x+6})} = \\ &= \lim_{x \rightarrow 1} \frac{\frac{\sqrt{3x+2} - \sqrt{2x+3}}{\cos\sqrt{3x+2} \cdot \cos\sqrt{2x+3}} \cdot \frac{\sin(\sqrt{3x+2} - \sqrt{2x+3})}{\sqrt{3x+2} - \sqrt{2x+3}} \cdot \frac{\sqrt{4x+3} - \sqrt{3x+4}}{\cos\sqrt{4x+3} \cdot \cos\sqrt{3x+4}} \cdot \frac{\sin(\sqrt{4x+3} - \sqrt{3x+4})}{\sqrt{4x+3} - \sqrt{3x+4}}}{\frac{\sqrt{5x+4} - \sqrt{4x+5}}{\cos\sqrt{5x+4} \cdot \cos\sqrt{4x+5}} \cdot \frac{\sin(\sqrt{5x+4} - \sqrt{4x+5})}{\sqrt{5x+4} - \sqrt{4x+5}} \cdot \frac{\sqrt{6x+5} - \sqrt{5x+6}}{\cos\sqrt{6x+5} \cdot \cos\sqrt{5x+6}} \cdot \frac{\sin(\sqrt{6x+5} - \sqrt{5x+6})}{\sqrt{6x+5} - \sqrt{5x+6}}} \\ &= \lim_{x \rightarrow 1} \frac{\frac{x-1}{(\sqrt{3x+2} + \sqrt{2x+3})\cos^2\sqrt{5}} \cdot \frac{x-1}{(\sqrt{4x+3} + \sqrt{3x+4})\cos^2\sqrt{7}}}{\frac{x-1}{(\sqrt{5x+4} + \sqrt{4x+5})\cos^2\sqrt{3}} \cdot \frac{x-1}{(\sqrt{6x+5} + \sqrt{5x+6})\cos^2\sqrt{11}}} = \end{aligned}$$

$$= 3 \sqrt{\frac{11}{35} \left(\frac{\cos 3 \cdot \cos \sqrt{11}}{\cos \sqrt{5} \cdot \cos \sqrt{7}} \right)^2}$$

Solution 4 by Abner Chinga Bazo-Peru

$$\begin{aligned} \Omega &= \lim_{x \rightarrow 1} \frac{(\tan \sqrt{3x+2} - \tan \sqrt{2x+3})(\tan \sqrt{4x+3} - \tan \sqrt{3x+4})}{(\tan \sqrt{5x+4} - \tan \sqrt{4x+5})(\tan \sqrt{6x+5} - \tan \sqrt{5x+6})} = \\ &= \lim_{x \rightarrow 1} \frac{\tan \sqrt{3x+2} - \tan \sqrt{2x+3}}{\tan \sqrt{5x+4} - \tan \sqrt{4x+5}} \cdot \lim_{x \rightarrow 1} \frac{\tan \sqrt{4x+3} - \tan \sqrt{3x+4}}{\tan \sqrt{6x+5} - \tan \sqrt{5x+6}} \stackrel{L'H}{=} \\ &\stackrel{L'H}{=} \lim_{x \rightarrow 1} \frac{\frac{3\sec^2 \sqrt{3x+2}}{2\sqrt{3x+2}} - \frac{\sec^2 \sqrt{2x+3}}{\sqrt{2x+3}}}{\frac{5\sec^2 \sqrt{5x+4}}{2\sqrt{5x+4}} - \frac{2\sec^2 \sqrt{4x+5}}{\sqrt{4x+5}}} \cdot \lim_{x \rightarrow 1} \frac{\frac{2\sec^2 \sqrt{4x+3}}{\sqrt{4x+3}} - \frac{3\sec^2 \sqrt{3x+4}}{2\sqrt{3x+4}}}{\frac{3\sec^2 \sqrt{6x+5}}{\sqrt{6x+5}} - \frac{5\sec^2 \sqrt{5x+6}}{2\sqrt{5x+6}}} \\ &= \frac{\frac{3\sec^2 \sqrt{5}}{2\sqrt{5}} - \frac{\sec^2 \sqrt{5}}{\sqrt{5}}}{\frac{5\sec^2 3}{2 \cdot 3} - \frac{2\sec^2 3}{3}} \cdot \frac{\frac{2\sec^2 \sqrt{7}}{\sqrt{7}} - \frac{3\sec^2 \sqrt{5}}{2\sqrt{7}}}{\frac{3\sec^2 \sqrt{11}}{\sqrt{11}} - \frac{5\sec^2 \sqrt{11}}{2\sqrt{11}}} \\ &= 3 \sqrt{\frac{11}{35} \left(\frac{\cos 3 \cdot \cos \sqrt{11}}{\cos \sqrt{5} \cdot \cos \sqrt{7}} \right)^2} \end{aligned}$$

1193. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(4^{-n} \sin \frac{1}{n} \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m} \right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\begin{aligned} \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m} &\leq \sum_{k=1}^{n+1} \sum_{m=0}^k \binom{n+1}{k} \binom{n}{m} = \\ &= \left(\sum_{k=1}^{n+1} \binom{n+1}{k} \right) \left(\sum_{m=0}^n \binom{n}{m} \right) = (2^{n+1} - 1) \cdot 2^n \end{aligned}$$

Let $a_n = 4^{-n} \sin \frac{1}{n} \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m}$ then $0 < a_n < 4 \sin \frac{1}{n}$, $\forall n \geq 1$



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$$\lim_{n \rightarrow \infty} 4 \sin \frac{1}{n} = 0 \Rightarrow$$

$$\Omega = \lim_{n \rightarrow \infty} \left(4^{-n} \sin \frac{1}{n} \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m} \right) = 0$$

Solution by Remus Florin Stanca-Romania

$$\begin{aligned}
\Omega &= \lim_{n \rightarrow \infty} \left(4^{-n} \sin \frac{1}{n} \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m} \right) = \\
&= \lim_{n \rightarrow \infty} \left(\frac{4^{-n}}{n} \cdot \frac{\sin \frac{1}{n}}{\frac{1}{n}} \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m} \right) = \\
&= \lim_{n \rightarrow \infty} \left(\frac{1}{n \cdot 4^n} \cdot \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m} \right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{\sum_{k=1}^n \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m} + \sum_{m=0}^n \binom{n+1}{m+1} \binom{n}{m}}{n \cdot 4^n} \right) = \\
&= \lim_{n \rightarrow \infty} \left(\frac{\sum_{k=1}^n \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m}}{n \cdot 4^n} \right) + \lim_{n \rightarrow \infty} \left(\frac{\sum_{m=0}^n \binom{n+1}{m+1} \binom{n}{m}}{n \cdot 4^n} \right); \quad (1) \\
&\lim_{n \rightarrow \infty} \left(\frac{\sum_{k=1}^n \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m}}{n \cdot 4^n} \right) = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \binom{n+1}{k} \sum_{m=0}^{k-1} \binom{n}{m}}{n \cdot 4^n} = \\
&= \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \binom{n+1}{n} \left(2^n - \left(\binom{n}{k} + \dots + \binom{n}{k} \right) \right)}{n \cdot 4^n} = \\
&= \lim_{n \rightarrow \infty} \frac{2^n \sum_{k=1}^n \binom{n+1}{k} - \sum_{k=1}^n \binom{n+1}{k} \left(\binom{n}{k} + \dots + \binom{n}{k} \right)}{n \cdot 4^n} = \\
&= \lim_{n \rightarrow \infty} \frac{2^n (2^n - 2) - \sum_{k=1}^n \binom{n+1}{k} \left(\binom{n}{k} + \dots + \binom{n}{k} \right)}{n \cdot 4^n} = \\
&= - \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \binom{n+1}{k} \left(\binom{n}{k} + \dots + \binom{n}{k} \right)}{n \cdot 4^n} = - \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \left(\binom{n}{k} + \binom{n}{k-1} \right) \left(\binom{n}{k} + \dots + \binom{n}{k} \right)}{n \cdot 4^n} = \\
&= - \lim_{n \rightarrow \infty} \frac{1}{n \cdot 4^n} \left(\sum_{k=1}^n \binom{n}{k} \left(\binom{n}{k} + \dots + \binom{n}{k} \right) + \sum_{k=1}^n \binom{n}{k-1} \left(\binom{n}{k} + \dots + \binom{n}{k} \right) \right) =
\end{aligned}$$



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$$= -\lim_{n \rightarrow \infty} \frac{1}{n \cdot 4^n} \left[\left(\binom{n}{1} + \cdots + \binom{n}{n} \right)^2 + \left(\binom{n}{1} + \cdots + \binom{n}{n} \right) \right] = \\ = -\lim_{n \rightarrow \infty} \frac{(2^n - 1)^2 + 2^n - 1}{n \cdot 4^n} = 0; \quad (2)$$

$$\lim_{n \rightarrow \infty} \left(\frac{\sum_{m=0}^n \binom{n+1}{m+1} \binom{n}{m}}{n \cdot 4^n} \right) = \lim_{n \rightarrow \infty} \frac{2^n}{n \cdot 4^n} = 0; \quad (3)$$

From (1), (2), (3) we get:

$$\Omega = \lim_{n \rightarrow \infty} \left(4^{-n} \sin \frac{1}{n} \sum_{k=1}^{n+1} \sum_{m=0}^{k-1} \binom{n+1}{k} \binom{n}{m} \right) = 0$$

1194. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} \right)^n$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Asmat Qatea-Kabul-Afghanistan

We know: $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$ and $\lim_{n \rightarrow 0} \frac{a^n - 1}{n} = \log a$

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} \right)^n = \\ = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1 \right)^{\frac{\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1}{\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1} \cdot n} =$$



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$$\begin{aligned}
 & \lim_{n \rightarrow \infty} n \left(\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1 \right)^{\frac{1}{n}} = \\
 &= e^{\lim_{n \rightarrow 0} \frac{1}{n} \left[\frac{1}{2} (1+n)^n + \frac{1}{2} (1-n)^n - 1 \right]} = \\
 &= e^{\frac{1}{2} \lim_{n \rightarrow 0} \frac{(1+n)^n - 1}{n} + \frac{1}{2} \lim_{n \rightarrow 0} \frac{(1-n)^n - 1}{n}} = e^{\frac{1}{2} \lim_{n \rightarrow 0} (\log(1+n) + \log(1-n))} = e^0 = 1
 \end{aligned}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} \right)^n = 1$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} = e^{\frac{1}{n} \log \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)}; \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1 \text{ and } \lim_{n \rightarrow 0} \frac{a^n - 1}{n} = \log a$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} = 1$$

$$\left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} = e^{\frac{1}{n} \log \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)}; \lim_{n \rightarrow \infty} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} = 1$$

$$u = \left(\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} \right)^n =$$

$$\begin{aligned}
 & \frac{\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1}{\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1} \cdot n \\
 &= \left(1 + \frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1 \right)^{\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1}
 \end{aligned}$$



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$$t_n := \frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1$$

$$\lim_{n \rightarrow \infty} \left[(1 + t_n)^{\frac{n}{t_n}} \right]^{t_n} = \lim_{n \rightarrow \infty} \left[(1 + t_n)^{\frac{1}{t_n}} \right]^{n \cdot t_n}$$

Let find $\lim_{n \rightarrow \infty} n \cdot t_n$

$$\mu(n) = n \left(\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1 \right) =$$

$$= \frac{\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} - 1}{\frac{1}{n}} = \frac{1}{2} \cdot \frac{\left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}}}{\frac{1}{n}} + \frac{1}{2} \cdot \frac{\left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}}}{\frac{1}{n}} =$$

$$= \frac{1}{2} \log \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} \cdot \frac{e^{\frac{1}{n} \log \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)} - 1}{\frac{1}{n} \log \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)} + \frac{1}{2} \log \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} \cdot \frac{e^{\frac{1}{n} \log \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)} - 1}{\frac{1}{n} \log \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)} =$$

$$= \frac{1}{2} \log 1 + \frac{1}{2} \log 1 = 0. \text{ Therefore,}$$

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{2} \left(1 + \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} + \frac{1}{2} \left(1 - \frac{n^{\frac{1}{n}}}{n} \right)^{\frac{1}{n}} \right)^n = 1$$

Solution 3 by Naren Bhandari-Bajura-Nepal

For $n \in \mathbb{N}$, it is easy to verify that $n \leq n! \leq n^n$ holds true and taking n -th root we have

$$\sqrt[n]{n} \leq \sqrt[n]{n!} \leq n \text{ which implies } \frac{\sqrt[n]{n}}{n} \leq 1; \quad (1)$$

$$F_b(n) = \frac{1}{b} \left(1 \pm \frac{\sqrt[n]{n}}{n} \right)^{\frac{1}{n}} = \frac{1}{b} \sum_{r=0}^{\infty} \binom{1}{r} \left(\pm \frac{\sqrt[n]{n}}{n} \right)^r$$



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since for all $n > 1$, $|x| < 1$ where $x = \pm \frac{\sqrt[n]{n}}{n}$ and hence by fractional binomial theorem we

write

$$F_b(n) = 1 \pm \frac{\sqrt[n]{n}}{n^2} \pm \frac{(1-n)\sqrt[n]{n^2}}{2! n^3} \pm \frac{(1-n)(1-2n)\sqrt[n]{n^2}}{3! n^4} \pm \dots; \quad (1)$$

From (2) observe that

$$\begin{aligned} \frac{1}{b} \sum_{r=0}^{\infty} \binom{\frac{1}{n}}{r} \left(\frac{\sqrt[n]{n}}{n}\right)^r + \frac{1}{b} \sum_{r=0}^{\infty} \binom{\frac{1}{n}}{r} \left(-\frac{\sqrt[n]{n}}{n}\right)^r &= \frac{1}{b} \left(2 + 2 \sum_{r=0}^{\infty} \binom{\frac{1}{n}}{2r} \left(\frac{\sqrt[n]{n}}{n}\right)^{2r} \right) = \\ &= \frac{2}{b} (1 + R(n)) \text{ which tends to } \frac{2}{b} \text{ as } R(n) \xrightarrow{n \rightarrow \infty} 0 \text{ and we have} \end{aligned}$$

$$\lim_{n \rightarrow \infty} F_n(2) = 1^\infty$$

From and heading to the main problem we have

$$\Omega = \lim_{n \rightarrow \infty} (1 + R(n))^n = \exp \left(\lim_{n \rightarrow \infty} R(n) \right) = e^0 = 1$$

1195.

$$u_n = \log \sqrt[3]{\frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{n^4 + 14n^3 + 73n^2 + 168n + 144}}$$

$S_n = u_1 + u_2 + \dots + u_n$. Find:

$$\Omega = \lim_{n \rightarrow \infty} n \left(\frac{3}{2} S_n - 4 \log n - \log 72 \right)$$

Proposed by Costel Florea-Romania

Solution by Adrian Popa-Romania

$$n^4 + 6n^3 + 13n^2 + 12n + 4 = (n+1)^2(n+2)^2$$

$$n^4 + 14n^3 + 73n^2 + 168n + 144 = (n+3)^2(n+4)^2$$

$$u_n = \frac{2}{3} \log \frac{(n+1)(n+2)}{(n+3)(n+4)} \Rightarrow S_n = \frac{2}{3} \log (72n(n+1)^2(n+2))$$

$$\Omega = \lim_{n \rightarrow \infty} n \left(\frac{2}{3} \cdot \frac{3}{2} \cdot \log (72n(n+1)^2(n+2)) - \log n^4 - \log 72 \right) =$$

$$= \lim_{n \rightarrow \infty} n \left(\log (72n(n+1)^2(n+2)) - \log (72n^4) \right) =$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} n \log \frac{n^3 + 4n^2 + 5n + 2}{n^3} = \log \left(\lim_{n \rightarrow \infty} \left(1 + \frac{4n^2 + 5n + 2}{n^3} \right)^n \right) = \\
 &= \log \left(e^{\lim_{n \rightarrow \infty} n \frac{4n^2 + 5n + 2}{n^3}} \right) = 4
 \end{aligned}$$

1196.

$$\Omega_1(n) = \lim_{x \rightarrow 0} \left(x \cdot \frac{\sum_{k=1}^n \sin(k(k+1)x)}{\sum_{k=1}^n \tan((\sin^{-1}(kx))^2)} \right)$$

Find:

$$\Omega = \lim_{n \rightarrow \infty} \sqrt[3]{(\Omega_1(n))^{2n}}$$

Proposed by Costel Florea-Romania

Solution 1 by Adrian Popa-Romania

$$\begin{aligned}
 \Omega_1(n) &= \lim_{x \rightarrow 0} \left(x \cdot \frac{\sum_{k=1}^n \sin(k(k+1)x)}{\sum_{k=1}^n \tan((\sin^{-1}(kx))^2)} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{x^2}{x} \cdot \frac{\sum_{k=1}^n \sin(k(k+1)x)}{\sum_{k=1}^n \tan\left(\left(\frac{\sin^{-1}(kx)}{x}\right)^2 x^2\right)} \right) = \\
 &= \lim_{x \rightarrow 0} \frac{\sum_{k=1}^n \frac{\sin(k(k+1)x)}{x}}{\sum_{k=1}^n \frac{\tan^2 x^2}{x^2}} = \frac{\sum_{k=1}^n k(k+1)}{\sum_{k=1}^n k^2} = 1 + \frac{3}{2n+1} \\
 \Omega &= \lim_{n \rightarrow \infty} \sqrt[3]{(\Omega_1(n))^{2n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{2n+1} \right)^{\frac{2n}{3}} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{3}{2n+1} \right)^{\frac{2n+1}{2}} \right]^{\frac{2n}{2n+1}} = e
 \end{aligned}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1; \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$



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$$\sum_{k=1}^n \sin(k(k+1)x) = \sum_{k=1}^n k(k+1)x \cdot \frac{\sin(k(k+1)x)}{k(k+1)x} = x \sum_{k=1}^n k(k+1)$$

$$\begin{aligned} \sum_{k=1}^n \tan\left(\left(\sin^{-1}(kx)\right)^2\right) &= \sum_{k=1}^n \frac{\tan\left(\left(\sin^{-1}(kx)\right)^2\right)}{\left(\sin^{-1}(kx)\right)^2} \cdot \frac{\left(\sin^{-1}(kx)\right)^2}{(kx)^2} \cdot k^2 x^2 = \\ &= x^2 \sum_{k=1}^n k^2 \end{aligned}$$

$$\Omega_1(n) = \frac{\sum_{k=1}^n k(k+1)}{\sum_{k=1}^n k^2} = 1 + \frac{3}{2n+1}$$

$$\Omega = \lim_{n \rightarrow \infty} \sqrt[3]{(\Omega_1(n))^{2n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{2n+1}\right)^{\frac{2n}{3}} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{3}{2n+1}\right)^{\frac{2n+1}{2}}\right]^{\frac{2n}{2n+1}} = e$$

1197. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{3^n((2n-1)!! + (2n)!!)}{H_{n+1}(2n^2+1)^n + H_n(2n^2+6n+4)^n} \right)$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

$$a_n = \frac{3^n((2n-1)!! + (2n)!!)}{H_{n+1}(2n^2+1)^n + H_n(2n^2+6n+4)^n}$$

For $n \geq 1$, $(2n-1)!! \leq (2n)^n$ and $(2n)!! \leq (2n)^n$

Also, $(2n^2+1)^n > 2^n n^{2n}$, $(2n^2+6n+4)^n > 2^n n^{2n}$ and $H_n \geq 1$

Thus,

$$0 < \frac{3^n((2n-1)!! + (2n)!!)}{H_{n+1}(2n^2+1)^n + H_n(2n^2+6n+4)^n} \leq \frac{3^n(2n)^n + 3^n(2n)^n}{2^n n^{2n} + 2^n n^{2n}} = \frac{2(3^n 2^n n^n)}{2 \cdot 2^n n^{2n}}$$

$$0 \leq a_n \leq \left(\frac{3}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(\frac{3}{n}\right)^n = 0$$



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Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{3^n((2n-1)!! + (2n)!!)}{H_{n+1}(2n^2+1)^n + H_n(2n^2+6n+4)^n} \right) = 0$$

1198. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \exp \left(\frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) - n \right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Arghyadeep Chatterjee-Kolkata-India

$$\begin{aligned} \exp \left(\frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) &= 1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} + \frac{n^2 \left(\tan^{-1} \left(\frac{k}{n} \right) \right)^2}{(k^2 + n^2)^2} \cdot \frac{1}{2!} + \dots \\ \sum_{k=1}^n \exp \left(\frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} \right) - n &= \sum_{k=1}^n \left(1 + \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} + \frac{n^2 \left(\tan^{-1} \left(\frac{k}{n} \right) \right)^2}{(k^2 + n^2)^2} \cdot \frac{1}{2!} + \dots \right) - n \\ &= \sum_{k=1}^n \frac{n \tan^{-1} \left(\frac{k}{n} \right)}{k^2 + n^2} + \sum_{k=1}^n \frac{n^2 \left(\tan^{-1} \left(\frac{k}{n} \right) \right)^2}{(k^2 + n^2)^2} \cdot \frac{1}{2!} + \dots \\ &= \frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1} \left(\frac{k}{n} \right)}{1 + \left(\frac{k}{n} \right)^2} + \frac{1}{n} \cdot \frac{1}{2!} \cdot \frac{1}{n} \sum_{k=1}^n \frac{\left(\tan^{-1} \left(\frac{k}{n} \right) \right)^2}{\left(1 + \left(\frac{k}{n} \right)^2 \right)^2} \end{aligned}$$

Using Riemann sums when $n \rightarrow \infty$ we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx + \frac{1}{2n} \int_0^1 \left(\frac{\tan^{-1} x}{1+x^2} \right)^2 dx + o \left(\frac{1}{n^2} \right) \right) &= \\ &= \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx = \frac{\pi^2}{32} \end{aligned}$$



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Solution 2 by Ali Jaffal-Lebanon

$$\text{Let } f(x) = \begin{cases} \frac{e^x - x - 1}{x^2}; & x \neq 0 \\ \frac{1}{2}; & x = 0 \end{cases}$$

When $x \in [0, 1]$ we have:

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}e^x + x^2 \varepsilon(x)}{x^2} = \\ &= \lim_{x \rightarrow 0} \frac{x^2 \left(\frac{1}{2} + \varepsilon(x) \right)}{x^2} = \lim_{x \rightarrow 0} \left(\frac{1}{2} + \varepsilon(x) \right) = \frac{1}{2} = f(0) \end{aligned}$$

So, f is continuous at $x = 0$.

$$\text{Let } x_n^{(k)} = \frac{\tan^{-1}\left(\frac{k}{n}\right)}{n\left(1 + \left(\frac{k}{n}\right)^2\right)}; \quad 1 \leq k \leq n, n \in \mathbb{N}^*$$

We have: $1 + \frac{1}{n^2} \leq 1 + \left(\frac{k}{n}\right)^2 \leq 2$, since $1 \leq k \leq n$ hence,

$$\frac{1}{2n} < \frac{1}{n\left(1 + \left(\frac{k}{n}\right)^2\right)} < \frac{n}{1 + n^2} \leq \frac{1}{n}$$

$$0 \leq \tan^{-1}\left(\frac{k}{n}\right) \leq \frac{\pi}{4} \Rightarrow 0 \leq \frac{\tan^{-1}\left(\frac{k}{n}\right)}{n\left(1 + \left(\frac{k}{n}\right)^2\right)} \leq \frac{\pi}{4n} \leq 1$$

Hence, $0 \leq x_n^{(k)} \leq 1$ for all $n \in \mathbb{N}^*$.

We know that f is continuous on $[0, 1]$ then exist $M \in \mathbb{R}_+$ such that

$$|f(x)| \leq M, \forall x \in [0, 1]$$

We have: $e^x = x^2 f(x) + x + 1, \forall x \in [0, 1]$ so,

$$e^{x_n} = x_n^2 f(x_n) + x_n + 1, \forall n \in \mathbb{N}^*$$

$$\left| \sum_{k=1}^n x_n^2 f(x_n(k)) \right| \leq M \sum_{k=1}^n x_n^2(k) \leq M \sum_{k=1}^n \frac{\left(\tan^{-1}\left(\frac{k}{n}\right) \right)^2}{n^2 \left(1 + \left(\frac{k}{n}\right)^2 \right)^2}$$



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$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\left(\tan^{-1}\left(\frac{k}{n}\right)\right)^2}{n^2 \left(1 + \left(\frac{k}{n}\right)^2\right)^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{1}{n} \sum_{k=1}^n \frac{\left(\tan^{-1}\left(\frac{k}{n}\right)\right)^2}{\left(1 + \left(\frac{k}{n}\right)^2\right)^2} = 0$$

Hence,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n x_n^2 f(x_n(k)) = 0$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n x_n(k) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1}\left(\frac{k}{n}\right)}{1 + \left(\frac{k}{n}\right)^2} = \int_0^1 \frac{\tan^{-1}x}{1+x^2} dx = \frac{1}{2} (\tan^{-1}x)^2 \Big|_0^1 = \frac{\pi^2}{32}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \exp\left(\frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2}\right) - n \right) = \frac{\pi^2}{32}$$

Solution 3 by Tobi Joshua-Nigeria

$$\begin{aligned} \Omega &= \sum_{k=1}^n (e^{f(n)} - n) = \sum_{k=1}^n n \left(\frac{e^{f(n)}}{n} - 1 \right) = \sum_{k=1}^n n \left(1 + \frac{f(n)}{n} + o\left(\frac{1}{n^2}\right) - 1 \right) = \\ &= \sum_{k=1}^n n \left(\frac{f(n)}{n} \right) = \sum_{k=1}^n f(n) \end{aligned}$$

$$\text{Replace } f(n) = \frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2}$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2} = \int_0^1 \frac{\tan^{-1}x}{1+x^2} dx = \frac{1}{2} (\tan^{-1}x)^2 \Big|_0^1 = \frac{\pi^2}{32}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \exp\left(\frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2}\right) - n \right) = \frac{\pi^2}{32}$$

Solution 4 by Florică Anastase-Romania

$$\lim_{t \rightarrow 0} \frac{e^t - 1}{t} = 1 \Rightarrow \forall \varepsilon > 0, \exists n_\varepsilon > 0 \text{ such that for all } n > n_\varepsilon \text{ we have:}$$



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$$1 - \varepsilon \leq \frac{e^t - 1}{t} \leq 1 + \varepsilon, \forall t > 0 \Leftrightarrow \\ (1 - \varepsilon)t \leq e^t - 1 \leq (1 + \varepsilon)t \Leftrightarrow$$

$$\sum_{k=1}^n \exp\left(\frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2}\right) - n = \sum_{k=1}^n \left[\exp\left(\frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2}\right) - 1 \right] \cong \sum_{k=1}^n \frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2}; \quad (1)$$

Let $f: [0, 1] \rightarrow \mathbb{R}$, $f(x) = \frac{\tan^{-1}x}{1+x^2}$, $x_n^k = \bar{z}_k = \frac{k}{n}$; $\|\Delta_n\| \rightarrow 0$, hence,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\tan^{-1}\left(\frac{k}{n}\right)}{1 + \left(\frac{k}{n}\right)^2} = \\ = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(\frac{k}{n}\right) \left(\frac{k}{n} - \frac{k-1}{n}\right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(\bar{z}_k^n) (x_k^n - x_k^{n-1}) = \\ = \int_0^1 \frac{\tan^{-1}x}{1+x^2} dx = \frac{1}{2} (\tan^{-1}x)^2 \Big|_0^1 = \frac{\pi^2}{32}; \quad (2)$$

From (1), (2) we get:

$$\Omega = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \exp\left(\frac{n \tan^{-1}\left(\frac{k}{n}\right)}{k^2 + n^2}\right) - n \right) = \frac{\pi^2}{32}$$

1199. If $0 < a \leq b < 1$, $f: [0, 1] \rightarrow [0, 1]$, f –continuous. Prove that:

$$3(b-a)^2 \int_a^b f^2(x) dx \leq 2(b-a)^3 + \left(\int_a^b f(x) dx \right)^3$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$2(b-a)^3 + \left(\int_a^b f(x) dx \right)^3 \stackrel{AGM}{\geq} 3 \sqrt[3]{(b-a)^6 \left(\int_a^b f(x) dx \right)^3} =$$



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$$= 3(b-a)^2 \int_a^b f(x) dx \geq 3(b-a)^2 \int_a^b f^2(x) dx$$

$$f(x) \in [0, 1] \Rightarrow f(x) \geq f^2(x)$$

1200.

$$\sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} + L_{2n+1}}{(2n+1)\phi^{4n}} = \frac{\pi(\phi+1)}{4\sqrt{5}} + \phi^2 \coth^{-1}(\sqrt{5})$$

ϕ – Golden Ratio, F_n – Fibonacci number, L_n – Lucas number

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Dawid Bialek-Poland

Note $F_{2n+1} := \frac{\varphi^{2n+1}}{\sqrt{5}} - \frac{(-\varphi)^{-2n-1}}{\sqrt{5}}$; (*), $L_{2n+1} := \varphi^{2n+1} + (-\varphi)^{-2n-1}$; (**)

So:

$$\Omega = \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}}{(2n+1)\phi^{4n}} + \sum_{n=0}^{\infty} \frac{L_{2n+1}}{(2n+1)\phi^{4n}} = S_1 + S_2; \quad (1)$$

$$S_1 \stackrel{(*)}{=} \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\phi^{4n}} (\varphi^{2n+1} - (-\varphi)^{-2n-1}) =$$

$$= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{2n+1-4n}}{(2n+1)} + \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{-2n-1-4n}}{(2n+1)} =$$

$$= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{-2n+1}}{(2n+1)} + \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{-6n-1}}{(2n+1)}$$

$$S_1 = \frac{\varphi^2}{\sqrt{5}} \left(\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\varphi}\right)^{2n+1}}{2n+1}}_{:= \tan^{-1}\left(\frac{1}{\varphi}\right)} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\varphi^3}\right)^{2n+1}}{2n+1}}_{:= \tan^{-1}\left(\frac{1}{\varphi^3}\right)} \right) =$$

$$= \frac{\varphi^2}{\sqrt{5}} \left(\tan^{-1}\left(\frac{1}{\varphi}\right) + \tan^{-1}\left(\frac{1}{\varphi^3}\right) \right) = \frac{\varphi^2}{\sqrt{5}} \tan^{-1} \left(\frac{\frac{1}{\varphi} + \frac{1}{\varphi^3}}{1 - \frac{1}{\varphi} \cdot \frac{1}{\varphi^3}} \right) =$$



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$$= \frac{\varphi^2}{\sqrt{5}} \tan^{-1} \left(\frac{\varphi(\varphi^2 + 1)}{\varphi^4 - 1} \right)$$

$$S_1 = \frac{\varphi^2}{\sqrt{5}} \tan^{-1} \left(\frac{\varphi}{\varphi^2 - 1} \right)^{\varphi^2 - 1 = \varphi} = \frac{\varphi^2}{\sqrt{5}} \tan^{-1}(1) = \frac{\varphi^2}{\sqrt{5}} \cdot \frac{\pi}{4}^{\varphi^2 = \varphi + 1} = \frac{\pi(\varphi + 1)}{4\sqrt{5}}$$

Now

$$\begin{aligned} S_2 &\stackrel{(**)}{=} \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\varphi^{4n}} (\varphi^{2n+1} + (-\varphi)^{-2n-1}) = \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{2n+1-4n}}{(2n+1)} - \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{-2n-1-4n}}{(2n+1)} = \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{-2n+1}}{(2n+1)} - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{-6n-1}}{(2n+1)} \\ S_2 &= \varphi^2 \left(\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\varphi}\right)^{2n+1}}{2n+1}}_{:= \tanh^{-1}\left(\frac{1}{\varphi}\right)} - \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\varphi^3}\right)^{2n+1}}{2n+1}}_{:= \tanh^{-1}\left(\frac{1}{\varphi^3}\right)} \right) = \end{aligned}$$

$$\begin{aligned} &= \frac{\varphi^2}{\sqrt{5}} \left(\tanh^{-1}\left(\frac{1}{\varphi}\right) - \tanh^{-1}\left(\frac{1}{\varphi^3}\right) \right) = \frac{\varphi^2}{\sqrt{5}} \tanh^{-1} \left(\frac{\frac{1}{\varphi} - \frac{1}{\varphi^3}}{1 + \frac{1}{\varphi} \cdot \frac{1}{\varphi^3}} \right) = \\ &= \frac{\varphi^2}{\sqrt{5}} \tanh^{-1} \left(\frac{\varphi}{\varphi^2 + 1} \right) \end{aligned}$$

$$\begin{aligned} S_2 &= \varphi^2 \tanh^{-1} \left(\frac{1 + \sqrt{5}}{5 + \sqrt{5}} \right) = \varphi^2 \tanh^{-1} \left(\frac{(1 + \sqrt{5})(5 - \sqrt{5})}{20} \right) = \\ &= \varphi^2 \tanh^{-1} \left(\frac{\sqrt{5}}{5} \right) = \varphi^2 \tanh^{-1} \left(\frac{1}{\sqrt{5}} \right) = \varphi^2 \coth^{-1}(\sqrt{5}) \end{aligned}$$

Rewriting (1) with S_1 & S_2 , we get:

$$\sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} + L_{2n+1}}{(2n+1)\varphi^{4n}} = \frac{\pi(\varphi+1)}{4\sqrt{5}} + \varphi^2 \coth^{-1}(\sqrt{5})$$



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It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru