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Find:

$$\Omega(n) = \lim_{x\to 0}\left(\frac{5^x-1}{x^{n+1}}-\frac{log5}{x^n}-\frac{log^2 5}{2x^{n-2}}-\dots-\frac{log^n 5}{n!\cdot x}\right), n\in\mathbb{N}, n\geq 2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Remus Florin Stanca-Romania, Solution 2 by Kaushik Mahanta-Assam-India

Solution 1 by Remus Florin Stanca-Romania

$$\Omega(n) = \lim_{x\to 0}\left(\frac{5^x-1}{x^{n+1}}-\frac{log5}{x^n}-\frac{log^2 5}{2x^{n-2}}-\dots-\frac{log^n 5}{n!\cdot x}\right) =$$

$$= \lim_{x\to 0}\frac{5^x-1-\frac{xlog5}{1!}-\frac{(xlog5)^2}{2!}-\dots-\frac{(xlog5)^n}{n!}}{x^{n+1}} =$$

$$= \lim_{x\to 0}\frac{5^x-\left(1+\frac{xlog5}{1!}+\frac{(xlog5)^2}{2!}+\dots+\frac{(xlog5)^n}{n!}\right)}{x^{n+1}};\ (1)$$

Let $f_n(x) = 1+\frac{xlog5}{1!}+\frac{(xlog5)^2}{2!}+\dots+\frac{(xlog5)^n}{n!}$ then,

$$\Omega(n) = \lim_{x\to 0}\frac{5^x-f_n(x)}{x^{n+1}} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{5^x log5-f'_n(x)}{(n+1)x^n} \overset{f'_n(x)=f_{n-1}(x)log5}{=}$$

$$= \lim_{x\to 0}\frac{(5^x - f_{n-1}(x))log5}{(n+1)x^n}$$

Let $a_n = \lim_{x\to 0}\frac{5^x - f_{n-1}(x)}{x^{n+1}} \Rightarrow a_n = \frac{a_{n-1}}{n+1}log5 \Rightarrow \frac{a_n}{a_{n-1}} = \frac{log5}{n+1} \Rightarrow$

$$\prod_{k=1}^{n}\frac{a_k}{a_{k-1}} = \frac{log^n 5}{(n+1)!} \Rightarrow \frac{a_n}{a_0} = \frac{log^n 5}{(n+1)!}$$

$$\lim_{x\to 0}\frac{5^x - f_{n-1}(x)}{x^{n+1}} = \lim_{x\to 0}\frac{5^x - 1}{x}\frac{log^n 5}{(n+1)!}$$

Therefore,

$$\Omega(n) = \lim_{x\to 0}\left(\frac{5^x - 1}{x^{n+1}} - \frac{log5}{x^n} - \frac{log^2 5}{2x^{n-2}} - \cdots - \frac{log^n 5}{n!\cdot x}\right) = \frac{log^{n+1}5}{(n+1)!}$$

Solution 2 by Kaushik Mahanta-Assam-India

$$\Omega(n) = \lim_{x\to 0}\left(\frac{5^x - 1}{x^{n+1}} - \frac{log5}{x^n} - \frac{log^2 5}{2x^{n-2}} - \cdots - \frac{log^n 5}{n!\cdot x}\right) =$$

$$= \lim_{x\to 0}\left(\frac{n!\,(5^x - 1) - n!\,xlog5 - \frac{n!}{2!}x^2 log^2 5 - \cdots - \frac{n!}{n!}x^n log^n 5}{n!\,x^{n+1}}\right) \overset{\frac{0}{0}}{=}$$

$$= \lim_{x\to 0}\left(\frac{n!\,(5^x log5) - n!\,log5 - \frac{n!}{1!}xlog^2 5 - \cdots - \frac{n!}{(n-1)!}x^{n-1}log^n 5}{n!\,(n+1)x^n}\right) =$$

$$= \lim_{x\to 0}\left(\frac{n!\,log5(5^x - 1) - \frac{n!}{1!}xlog^2 5 - \cdots - \frac{n!}{(n-1)!}x^{n-1}log^n 5}{n!\,(n+1)x^n}\right) \overset{\frac{0}{0}}{=}$$

$$= \lim_{x\to 0}\left(\frac{n!\,log^2 5 5^x - n!\,log^2 5 - \cdots - \frac{n!}{(n-1)!}x^{n-2}log^n 5}{n!\,(n+1)nx^{n-1}}\right) =$$

$$= \lim_{x\to 0}\left(\frac{n!\,log^2 5(5^x - 1) - \cdots - \frac{n!}{(n-2)!}x^{n-2}log^n 5}{n!\,(n+1)nx^{n-1}}\right) \overset{\frac{0}{0}}{=}$$

$$= \cdots = \lim_{x\to 0}\frac{n!\,5^x log^n 5 - n!\,log5}{n!\,(n+1)!\,x} =$$

$$= \lim_{x \to 0} \frac{n!\, log^n 5}{n!\,(n+1)!} \frac{5^x - 1}{x} = \frac{log^{n+1} 5}{(n+1)!}$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.