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Find:

$$\Omega=\lim_{n\to\infty}\sum_{1\le i<j\le n}\frac{\cos\left(\frac{j-i}{n}\right)-\cos\left(\frac{i+j}{n}\right)}{\sqrt{i^2+j^2+n^4}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florentin Vişescu-Romania, Solution 2 by Florică Anastase-Romania

Solution 1 by Florentin Vişescu-Romania

$$\cos\left(\frac{j-i}{n}\right)-\cos\left(\frac{i+j}{n}\right)=2\sin\frac{j}{n}\sin\frac{i}{n}$$

$$\sum_{1\le i<j\le n}\frac{\cos\left(\frac{j-i}{n}\right)-\cos\left(\frac{i+j}{n}\right)}{\sqrt{i^2+j^2+n^4}}=2\sum_{1\le i<j\le n}\frac{\sin\frac{j}{n}\sin\frac{i}{n}}{\sqrt{i^2+j^2+n^4}}$$

$$2\sum_{1\le i<j\le n}\frac{\sin\frac{j}{n}\sin\frac{i}{n}}{\sqrt{(n-1)^2+n^2+n^4}}\le 2\sum_{1\le i<j\le n}\frac{\sin\frac{j}{n}\sin\frac{i}{n}}{\sqrt{i^2+j^2+n^4}}\le 2\sum_{1\le i<j\le n}\frac{\sin\frac{j}{n}\sin\frac{i}{n}}{\sqrt{1^2+2^2+n^4}}$$

$$\frac{2}{\sqrt{(n-1)^2+n^2+n^4}}\sum_{1\le i<j\le n} sin\frac{j}{n}sin\frac{i}{n}\le 2\sum_{1\le i<j\le n}\frac{sin\frac{j}{n}sin\frac{i}{n}}{\sqrt{i^2+j^2+n^4}}$$

$$\le\frac{2}{\sqrt{1^2+2^2+n^4}}\sum_{1\le i<j\le n} sin\frac{j}{n}sin\frac{i}{n}$$

We want to find:

$$\lim_{n\to\infty}\frac{2}{n^2}\sum_{1\le i<j\le n} sin\frac{j}{n}sin\frac{i}{n}$$

$$\left(sin\frac{1}{n}+sin\frac{2}{n}+\cdots+sin\frac{1}{n}\right)^2=2\sum_{1\le i<j\le n} sin\frac{j}{n}sin\frac{i}{n}+\sum_{i=1}^{n} sin^2\frac{i}{n}$$

$$\frac{1}{n^2}\left(\sum_{i=1}^{n} sin\frac{i}{n}\right)^2=\frac{2}{n^2}\sum_{1\le i<j\le n} sin\frac{j}{n}sin\frac{i}{n}+\frac{1}{n^2}\sum_{i=1}^{n} sin^2\frac{i}{n}$$

$$\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} sin\frac{i}{n}\right)^2}_{\to\left(\int_0^1 sinxdx\right)^2}=\frac{2}{n^2}\sum_{1\le i<j\le n} sin\frac{j}{n}sin\frac{i}{n}+\frac{1}{n}\underbrace{\frac{1}{n}\sum_{i=1}^{n} sin^2\frac{i}{n}}_{\to\alpha\in\mathbb{R}}$$

$$\lim_{n\to\infty}\frac{2}{n^2}\sum_{1\le i<j\le n} sin\frac{j}{n}sin\frac{i}{n}=(cos1-cos0)^2=\left(-2sin^2\frac{1}{2}\right)^2=4sin^4\frac{1}{2}$$

$$\Omega=\lim_{n\to\infty}\sum_{1\le i<j\le n}\frac{cos\left(\frac{j-i}{n}\right)-cos\left(\frac{i+j}{n}\right)}{\sqrt{i^2+j^2+n^4}}=4sin^4\frac{1}{2}$$

Solution 2 by Florică Anastase-Romania

Lemma. If $f:[a,b]\to\mathbb{R}$ continuous function, then

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n} f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)=\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

Proof.

$$\left(\sum_{i=1}^{n} f(i)\right)^2=\sum_{i=1}^{n} f^2\left(\frac{i}{n}\right)+2\sum_{1\le i<j\le n} f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)\Leftrightarrow$$

$$\sum_{1\le i<j\le n} f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)=\frac{1}{2}\left[\left(\sum_{i=1}^{n} f(i)\right)^2-\sum_{i=1}^{n} f^2\left(\frac{i}{n}\right)\right]+\sum_{i=1}^{n} f^2\left(\frac{i}{n}\right)=$$

$$=\frac{1}{2}\left[\left(\sum_{i=1}^{n} f(i)\right)^2+\sum_{i=1}^{n} f^2\left(\frac{i}{n}\right)\right]$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n} f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)=\frac{1}{2}\lim_{n\to\infty}\frac{1}{n^2}\left[\left(\sum_{i=1}^{n} f(i)\right)^2+\sum_{i=1}^{n} f^2\left(\frac{i}{n}\right)\right]=$$

$$=\frac{1}{2}\lim_{n\to\infty}\left[\left(\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\right)^2+\underbrace{\frac{1}{n}\frac{1}{n}\sum_{i=1}^{n} f^2\left(\frac{i}{n}\right)}_{0}\right]=\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2$$

$$\cos\left(\frac{j-i}{n}\right)-\cos\left(\frac{i+j}{n}\right)=2\sin\frac{j}{n}\sin\frac{i}{n}$$

Let $f(t)=\sin t$

$$2\sum_{1\le i<j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+2n^2}}\le 2\sum_{1\le i<j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+i^2+j^2}}\le 2\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+n^2}}$$

$$\lim_{n\to\infty}\sum_{1\le i<j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+2n^2}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{1+\frac{2}{n^2}}}=\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2=$$

$$=\frac{1}{2}\left(\int_0^1 \sin x\,dx\right)^2=\frac{1}{2}(1-\cos 1)^2=\frac{1}{2}\left(2\sin^2\frac{1}{2}\right)^2=2\sin^4\frac{1}{2}$$

$$\lim_{n\to\infty}\sum_{1\le i<j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{n^4+n^2}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i\le j\le n}\frac{f\left(\frac{i}{n}\right)f\left(\frac{j}{n}\right)}{\sqrt{1+\frac{1}{n^2}}}=\frac{1}{2}\left(\int_0^1 f(x)dx\right)^2=2\sin^4\frac{1}{2}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{1\le i<j\le n}\frac{\cos\left(\frac{j-i}{n}\right)-\cos\left(\frac{i+j}{n}\right)}{\sqrt{i^2+j^2+n^4}}=4\sin^4\frac{1}{2}$$