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If $a_0 = r, a_{n+1} = a_n^2$ and $P_n(r) = \prod_{k=0}^{n}(1 + a_k)$ then find:

$$\lim_{n\to\infty}\int_0^1\left(P_n(r) - \frac{n+1}{r+1}\right)dr$$

Proposed by Jibran Iqbal-India

Solution by Abdul Hannan-Tezpur-India

If $a_0 = r, a_{n+1} = a_n^2$ then $a_n = r^{2^n}$

$$\Rightarrow P_n(r) = \prod_{k=0}^{n}\left(1 + r^{2^k}\right) = \prod_{k=0}^{n}\frac{1 - r^{2^{k+1}}}{1 - r^{2^k}} = \frac{1 - r^{2^{n+1}}}{1 - r} = \sum_{k=0}^{2^{n+1}-1} r^k$$

$$\int_0^1\left(P_n(r) - \frac{n+1}{r+1}\right)dr = \int_0^1\left(\sum_{k=0}^{2^{n+1}-1} r^k - \frac{n+1}{r+1}\right)dr = \sum_{k=0}^{2^{n+1}-1}\int_0^1 r^k\,dr - \int_0^1\frac{n+1}{r+1}dr =$$

$$= \sum_{k=0}^{2^{n+1}-1}\frac{1}{k+1} - (n+1)log2 = H_{2^{n+1}} - log2^{n+1}$$

Therefore, the desired answer is: $\lim_{n\to\infty}\left(H_{2^{n+1}} - log2^{n+1}\right) = \gamma$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.