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Find:

$$\Omega=\lim_{n\to\infty}\left|\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\sin\left(\frac{k}{n}\right)-\sin x\right|, x\in(0,1)$$

Proposed by Daniel Sitaru-Romania

Solution by Arghyadeep Chatterjee-Kolkata-India

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\sin\left(\frac{k}{n}\right)=I_n\left[\sum_{k=0}^{n}\binom{n}{k}x^k e^{\frac{ki}{n}}(1-x)^{n-k}\right]=$$

$$=I_n\left[\sum_{k=0}^{n}\binom{n}{k}\left(xe^{\frac{i}{n}}\right)^k(1-x)^{n-k}\right]=I_n\left(xe^{\frac{i}{n}}+(1-x)\right)^n$$

$$I_n\left[e^{\lim\limits_{n\to\infty}(xe^{\frac{i}{n}}-x)n}\right]=I_n\left[e^{\lim\limits_{n\to\infty}\frac{\left(e^{\frac{i}{n}}-1\right)x}{\frac{1}{n}}}\right]=\lim_{n\to\infty}I_n\left[e^{\frac{ix\left(e^{\frac{i}{n}}-1\right)}{\frac{i}{n}}}\right]=I_n e^{ix}=\sin x$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left|\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\sin\left(\frac{k}{n}\right)-\sin x\right|=0$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.