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If $f:[a,b]\to(0,\infty)$, f –continuous then:

$$2\int_a^b\int_a^b\int_a^b \frac{f^3(x)}{f^2(x)+f(y)f(z)}\,dx\,dy\,dz \ge (b-a)^2\int_a^b f(x)\,dx$$

Proposed by Daniel Sitaru-Romania

Solution by Florentin Vişescu-Romania

$$2\int_a^b\int_a^b\int_a^b \frac{f^3(x)}{f^2(x)+f(y)f(z)}\,dx\,dy\,dz =$$

$$= 2\int_a^b\int_a^b\int_a^b \frac{f^3(x)+f(x)f(y)f(z)-f(x)f(y)f(z)}{f^2(x)+f(y)f(z)}\,dx\,dy\,dz =$$

$$= 2\int_a^b\int_a^b\int_a^b f(x)f(y)f(z)dx\,dy\,dz - 2\int_a^b\int_a^b\int_a^b \frac{f(x)f(y)f(z)}{f^2(x)+f(y)f(z)}\,dx\,dy\,dz =$$

$$= 2(b-a)^2\int_a^b f(x)\,dx - 2\int_a^b\int_a^b\int_a^b \frac{f(x)f(y)f(z)}{f^2(x)+f(y)f(z)}\,dx\,dy\,dz;\quad (*)$$

$$\frac{f^2(x)+f(y)f(z)}{2} \ge f(x)\sqrt{f(y)f(z)} \Rightarrow \frac{1}{f^2(x)+f(y)f(z)} \le \frac{1}{2f(x)\sqrt{f(y)f(z)}}$$

$$\Rightarrow -\frac{2f(x)f(y)f(z)}{f^2(x)+f(y)f(z)} \geq -\frac{2f(x)f(y)f(z)}{2f(x)\sqrt{f(y)f(z)}} = -\sqrt{f(y)f(z)}$$

$$2(b-a)^2\int_a^b f(x)\,dx - 2\int_a^b\int_a^b\int_a^b \frac{f(x)f(y)f(z)}{f^2(x)+f(y)f(z)}dx\,dy\,dz \geq$$

$$\geq 2(b-a)^2\int_a^b f(x)dx - \int_a^b\int_a^b\int_a^b \sqrt{f(y)f(z)}dx\,dy\,dz =$$

$$= 2(b-a)^2\int_a^b f(x) - (b-a)\left(\int_a^b \sqrt{f(x)}dx\right)^2$$

We must show that:

$$2(b-a)^2\int_a^b f(x) - (b-a)\left(\int_a^b \sqrt{f(x)}dx\right)^2 \geq (b-a)^2\int_a^b f(x)\,dx$$

$$(b-a)\int_a^b f(x)\,dx \geq \left(\int_a^b \sqrt{f(x)}dx\right)^2 \text{ true from BCS:}$$

$$\left(\int_a^b \sqrt{f(x)}dx\right)^2 = \left(\int_a^b 1\cdot\sqrt{f(x)}dx\right)^2 \overset{BCS}{\leq} \int_a^b dx \cdot \int_a^b f(x)\,dx =$$

$$= (b-a)\int_a^b f(x)\,dx$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.