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If $f:[0,1]\to[0,\infty)$ ***such that*** $\int_0^1 f(x)dx = 1$,

$$I_1 = \int_0^1 \frac{x^2+x+1}{x^2+1}\cdot e^{\tan^{-1}x}dx,\ I_2 = \int_0^1\left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx$$

Then prove: $I_1 \ge e^{\pi I_2}$

Proposed by Florică Anastase-Romania

Solution 1 by Adrian Popa-Romania, Solution 2 by proposer

Solution 1 by Adrian Popa-Romania

$$I_1 = \int_0^1 \frac{x^2+x+1}{x^2+1}\cdot e^{\tan^{-1}x}dx \overset{\substack{\tan^{-1}x=t\\ dt=\frac{dx}{1+x^2}}}{=} \int_0^{\frac{\pi}{4}} (\tan^2 t + \tan t + 1)e^t\, dt$$

$$I_{11} = \int_0^{\frac{\pi}{4}} e^t \tan t\, dt \overset{IBP}{=} e^t \tan t\,\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^t(a+\tan^2 t)\, dt = e^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^t(1+\tan^2 t)\, dt$$

$$I_1 = I_{11} + \int_0^{\frac{\pi}{4}}(1+\tan^2 t)\, dt = e^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^t(1+\tan^2 t)\, dt + \int_0^{\frac{\pi}{4}} e^t(1+\tan^2 t)\, dt = e^{\frac{\pi}{4}}$$

We must to prove:

$$e^{\frac{\pi}{4}} \ge e^{\pi I_2} \Leftrightarrow I_2 \le \frac{1}{4}$$

$$I_2 = \int_0^1 x^2 f(x)\,dx - 2\int_0^1 xAf(x)\,dx + A^2\int_0^1 f(x)\,dx;$$

$$A = \int_0^1 tf(t)\,dt$$

$$I_2 = \int_0^1 x^2 f(x)\,dx - 2A\int_0^1 xf(x)\,dx + A^2\underbrace{\int_0^1 f(x)\,dx}_{=1} =$$

$$= \int_0^1 x^2 f(x)\,dx - 2\left(\int_0^1 xf(x)\,dx\right)\left(\int_0^1 xf(x)\,dx\right) + \left(\int_0^1 xf(x)\right)^2 =$$

$$= \int_0^1 x^2 f(x)\,dx - \left(\int_0^1 xf(x)\,dx\right)^2 \overset{(?)}{\le} \frac{1}{4} \Leftrightarrow$$

$$4\int_0^1 x^2 f(x)\,dx - 4\left(\int_0^1 xf(x)\,dx\right)^2 \le 1 \Leftrightarrow$$

$$4\left(\int_0^1 xf(x)\,dx\right)^2 - 4\int_0^1 x^2 f(x)\,dx + 1 \ge 0$$

$$x \in [0,1] \Rightarrow x^2 \le x \Rightarrow x^2 f(x) \le xf(x) \Rightarrow -x^2 f(x) \ge -xf(x)$$

$$4\left(\int_0^1 xf(x)\,dx\right)^2 - 4\int_0^1 x^2 f(x)\,dx + 1 \ge 4\left(\int_0^1 xf(x)\,dx\right)^2 - 4\int_0^1 xf(x)\,dx + 1 =$$

$$= \left(2\int_0^1 xf(x)\,dx - 1\right)^2 \ge 0$$

Solution 2 by proposer

$$I_1 = \int_0^1 \frac{x^2+x+1}{x^2+1}\cdot e^{\tan^{-1}x}dx = \int_0^1 \left(1+\frac{x}{1+x^2}\right)\cdot e^{\tan^{-1}x}dx =$$

$$= \int_0^1 e^{\tan^{-1}x}dx + \int_0^1 \frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx =$$

$$= x\cdot e^{\tan^{-1}x}\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx + \int_0^1 \frac{x}{1+x^2}\cdot e^{\tan^{-1}x}dx =$$

$$= x\cdot e^{\tan^{-1}x}\Big|_0^1 = e^{\frac{\pi}{4}};\ \ (1)$$

$$\int_0^1 \left(x-\frac{1}{2}\right)^2 f(x)dx = \int_0^1 \left(x-\int_0^1 tf(t)dt + \int_0^1 tf(t)dt - \frac{1}{2}\right)^2 f(x)dx =$$

$$= \int_0^1 \left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx + 2\int_0^1 \left(x-\int_0^1 tf(t)dt\right)\left(\int_0^1 tf(t)dt-\frac{1}{2}\right)f(x)dx +$$

$$+ \int_0^1 \left(\int_0^1 tf(t)dt-\frac{1}{2}\right)^2 f(x)dx =$$

$$= \int_0^1 \left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx + 2\left(\int_0^1 tf(t)dt-\frac{1}{2}\right)\int_0^1 \left(x-\int_0^1 tf(t)dt\right)f(x)dx +$$

$$+ \int_0^1 \left(\int_0^1 tf(t)dt-\frac{1}{2}\right)^2 f(x)dx \ge \int_0^1 \left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx$$

Hence,

$$I_2 = \int_0^1 \left(x-\int_0^1 tf(t)dt\right)^2 f(x)dx \le \int_0^1 \left(x-\frac{1}{2}\right)^2 f(x)dx \le \frac{1}{4}\int_0^1 f(x)dx = \frac{1}{4};\ \ (2)$$

From $(1),(2)$ ***we get:*** $I_1 \ge e^{\pi I_2}$