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$0 < a \le b, f, f'; [a,b] \to (0,\infty), f$ –nonconstant,derivable. Prove that:

$$6\int_a^b \frac{f'(x)}{\sqrt{1+f^2(x)}}dx + f^3(b) - f^3(a) \ge 6(f(b) - f(a))$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Nassim Nicholas Taleb-New York-USA, Solution 2 by proposer, Solution 3 by Florentin Vişescu-Romania

Solution 1 by Nassim Nicholas Taleb-New York-USA

$$6\int_a^b \frac{f'(x)}{\sqrt{1+f^2(x)}}dx = 6\int_{f(a)}^{f(b)} \frac{du}{\sqrt{1+u^2}} =$$

$$= 6log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right) \ge f^3(a) - f^3(b) + 6f(b) - 6f(a)$$

$$6\int_a^b \frac{f'(x)}{\sqrt{1+f^2(x)}}dx + f^3(b) - f^3(a) \ge 6(f(b) - f(a))$$

Solution 2 by proposer

$$g: [0,\infty) \to [0,\infty), g(x) = \frac{1}{\sqrt{1+x^2}}$$

$$g(x)\overset{MAC-LAURIN}{\triangleq} g(0)+\frac{x}{1!}g'(0)+\frac{x^2}{2!}g''(0)+\dots+\frac{x^{n-1}}{(n-1)!}g^{(n-1)}(0)$$

$$+\int_0^x\frac{(x-t)^{n-1}}{(n-1)!}g^{(n)}(t)dt\geq g(0)+\frac{x}{1!}g'(0)+\frac{x^2}{2!}g''(0)$$

$$g(0)=1, g'(0)=0, g''(0)=-1$$

$$\frac{1}{\sqrt{1+x^2}}\geq 1-\frac{x^2}{2}\to\frac{1}{\sqrt{1+f^2(x)}}\geq 1-\frac{f^2(x)}{2}\to$$

$$\frac{f'(x)}{\sqrt{1+f^2(x)}}\geq f'(x)-\frac{f^2(x)f'(x)}{2}$$

$$\int_a^b\frac{f'(x)}{\sqrt{1+f^2(x)}}dx\geq\int_a^b f'(x)dx-\int_a^b\frac{f^2(x)f'(x)}{2}dx=$$

$$=f(b)-f(a)-\frac{1}{6}(f^3(b)-f^3(a))$$

$$6\int_a^b\frac{f'(x)}{\sqrt{1+f^2(x)}}dx+f^3(b)-f^3(a)\geq 6(f(b)-f(a))$$

Solution 3 by Florentin Vişescu-Romania

$$6\int_a^b\frac{f'(x)}{\sqrt{1+f^2(x)}}dx+f^3(b)-f^3(a)\geq 6(f(b)-f(a))\Leftrightarrow$$

$$6log\left(f(x)+\sqrt{f^2(x)+1}\right)\Big|_a^b+f^3(b)-f^3(a)-6(f(b)-f(a))\geq 0\Leftrightarrow$$

$$6log\left(f(b)+\sqrt{f^2(b)+1}\right)+f^3(b)-6f(b)-6log\left(f(a)+\sqrt{f^2(a)+1}\right)-f^3(a)$$
$$+6f(a)\geq 0$$

Let $g:[a,b]\to\mathbb{R}, g(x)=6log\left(f(x)+\sqrt{f^2(x)+1}\right)+f^3(x)-6f(x)$

$$\overset{MVT}{\Longrightarrow} g'(x)=\frac{6f'(x)}{\sqrt{1+f^2(x)}}+3f^2(x)f'(x)-6f'(x)=$$

$$=\underbrace{f'(x)}_{>0}\left[\frac{6}{\sqrt{1+f^2(x)}}+3f^2(x)-6\right]$$

We must show that:

$$\frac{6}{\sqrt{1+\alpha^2}}+3\alpha-6>0, \forall \alpha>0$$

Let $h:(0,\infty)\to\mathbb{R}, h(\alpha)=\frac{6}{\sqrt{1+\alpha}}+3\alpha-6,$

$$h'(\alpha)=3\left(1-\frac{1}{(1+\alpha)\sqrt{1+\alpha}}\right)$$

$$h'(\alpha)=0\Rightarrow 1=(1+\alpha)\sqrt{1+\alpha}\Leftrightarrow(1+\alpha)^3=1\Rightarrow\alpha=0.$$

α	0 $\qquad\qquad\qquad\qquad$ ∞
$h'(\alpha)$	+ + + + + + + + + + + + + + + + + + +
$h(\alpha)$	0 ↗ ↗ ↗ ↗ ↗ ↗ ↗

So, $h(\alpha)>0, \forall\alpha>0.$

Note by editor:

Many thanks to Florica Anastase-Romania for typed solutions.