R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Let, for any complex number y

$$\int_{-\infty}^{\infty} \frac{e^{-\pi(x^2+xy)}}{(tanh(\pi x)+1)^2}dx = \psi(y)\int_{-\infty}^{\infty} \frac{e^{-\pi(x^2+xy)}}{(coth(\pi x)+1)^2}dx$$

Prove that:

$$\int_{-\infty}^{\infty} (\psi(y)-1)dy = \frac{4(\pi - sec^{-1}(e^{\pi}))}{\pi\sqrt{e^{2\pi}-1}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Mohammad Rostami-Kabul-Afganistan

$$\int_{-\infty}^{\infty} \frac{e^{-\pi(x^2+xy)}}{(tanh(\pi x)+1)^2}dx = \int_{-\infty}^{\infty} \frac{e^{-\pi x^2-\pi xy}}{(tanh(\pi x)+1)^2}dx =$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} (e^{4\pi x}+2e^{2\pi x}+1)e^{-\pi x^2-4\pi x-\pi xy}dx =$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} e^{-\pi x^2-\pi xy}dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-\pi x^2-2\pi x-\pi xy}dx + \frac{1}{4}\int_{-\infty}^{\infty} e^{-\pi x^2-4\pi x-\pi xy}dx =$$

$$= \frac{1}{4}e^{\frac{\pi}{4}y^2}\int_{-\infty}^{\infty} e^{-\pi\left(x+\frac{y}{2}\right)^2}dx + \frac{1}{2}e^{\frac{\pi}{4}(2+y)^2}\int_{-\infty}^{\infty} e^{-\pi\left(x+\frac{2+y}{2}\right)^2}dx + \frac{1}{4}e^{\frac{\pi}{4}(4+y)^2}\int_{-\infty}^{\infty} e^{-\pi\left(x+\frac{4+y}{2}\right)^2}dx$$

$$= \frac{1}{4\sqrt{\pi}}e^{\frac{\pi}{4}y^2}\int_{-\infty}^{\infty} e^{-t^2}dt + \frac{1}{2\sqrt{\pi}}e^{\frac{\pi}{4}(2+y)^2}\int_{-\infty}^{\infty} e^{-t^2}dt + \frac{1}{4\sqrt{\pi}}e^{\frac{\pi}{4}(4+y)^2}\int_{-\infty}^{\infty} e^{-t^2}dt$$

$$=\left(\frac{e^{\frac{\pi}{4}y^2}}{4\sqrt{\pi}}+\frac{e^{\frac{\pi}{4}(2+y)^2}}{2\sqrt{\pi}}+\frac{e^{\frac{\pi}{4}(4+y)^2}}{4\sqrt{\pi}}\right)\underbrace{\int_0^{\infty}u^{-\frac{1}{2}}e^{-u}du}_{\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}}=\frac{e^{\frac{\pi}{4}y^2}}{4}+\frac{e^{\frac{\pi}{4}(2+y)^2}}{2}+\frac{e^{\frac{\pi}{4}(4+y)^2}}{4}$$

$$\int_{-\infty}^{\infty}\frac{e^{-\pi(x^2+xy)}}{(coth(\pi x)+1)^2}dx=\int_{-\infty}^{\infty}\frac{e^{-\pi x^2-\pi xy}}{\left(\frac{2e^{2\pi x}}{e^{2\pi x}-1}\right)^2}dx=$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}(e^{4\pi x}-2e^{2\pi x}+1)e^{-\pi x^2-4\pi x-\pi xy}dx=$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}e^{-\pi x^2-\pi xy}dx-\frac{1}{2}\int_{-\infty}^{\infty}e^{-\pi x^2-2\pi x-\pi xy}dx+\frac{1}{4}\int_{-\infty}^{\infty}e^{-\pi x^2-4\pi x-\pi xy}dx=$$

$$=\frac{e^{\frac{\pi}{4}y^2}}{4}-\frac{e^{\frac{\pi}{4}(2+y)^2}}{2}+\frac{e^{\frac{\pi}{4}(4+y)^2}}{4}$$

$$\psi(y)=\frac{\frac{e^{\frac{\pi}{4}y^2}}{4}+\frac{e^{\frac{\pi}{4}(2+y)^2}}{2}+\frac{e^{\frac{\pi}{4}(4+y)^2}}{4}}{\frac{e^{\frac{\pi}{4}y^2}}{4}-\frac{e^{\frac{\pi}{4}(2+y)^2}}{2}+\frac{e^{\frac{\pi}{4}(4+y)^2}}{4}}=1+\frac{e^{\frac{\pi}{4}(2+y)^2}}{\frac{e^{\frac{\pi}{4}y^2}}{4}-\frac{e^{\frac{\pi}{4}(2+y)^2}}{2}+\frac{e^{\frac{\pi}{4}(4+y)^2}}{4}}$$

$$\psi(y)-1=\frac{4e^{\frac{\pi}{4}(2+y)^2}}{e^{\frac{\pi}{4}y^2}-2e^{\frac{\pi}{4}(2+y)^2}+e^{\frac{\pi}{4}(4+y)^2}}=\frac{4}{e^{-\pi y-\pi}-2+e^{\pi y+3\pi}}=$$

$$=\frac{4e^{\pi y}}{e^{-\pi}-2e^{\pi y}+e^{3\pi}e^{2\pi y}}$$

$$\frac{4}{\pi}\int_{-\infty}^{\infty}\frac{\pi e^{\pi y}}{e^{3\pi}(e^{\pi y})^2-2e^{\pi y}+e^{-\pi}}dy=\frac{4}{\pi}\int_0^{\infty}\frac{dz}{e^{3\pi}z^2-2z+e^{-\pi}}$$

$$\therefore\int\frac{dz}{az^2+bz+c}\overset{\Delta\le 0}{=}\frac{2}{\sqrt{4ac-b^2}}tan^{-1}\left(\frac{2az+b}{\sqrt{4ac-b^2}}\right)+C$$

$$\int_{-\infty}^{\infty}(\psi(y)-1)dy=\left[\frac{4}{\pi}\frac{2}{\sqrt{4e^{2\pi}-4}}tan^{-1}\left(\frac{2e^{3\pi}z-2}{\sqrt{4e^{2\pi}-4}}\right)\right]_0^{\infty}=$$

$$=\left[\frac{4}{\pi\sqrt{e^{2\pi}-1}}tan^{-1}\left(\frac{e^{3\pi}z-1}{\sqrt{e^{2\pi}-1}}\right)\right]_0^{\infty}=$$

$$=\frac{4}{\pi\sqrt{e^{2\pi}-1}}\left(\frac{\pi}{2}-tan^{-1}\left(-\frac{1}{\sqrt{(e^{\pi})^2-1}}\right)\right)=$$

$$=\frac{4}{\pi\sqrt{e^{2\pi}-1}}\left[\pi-\left(\frac{\pi}{2}+tan^{-1}\left(-\frac{1}{\sqrt{e^{2\pi}-1}}\right)\right)\right]=$$

$$=\frac{4}{\pi\sqrt{e^{2\pi}-1}}\left(\pi-sec^{-1}(e^{\pi})\right)$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.