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If $\psi(m,a) = \int_0^\infty \frac{e^{-x^2}x^{m-1}}{(1+x^2)^a}dx,\ a \in \mathbb{N}$ **then prove that:**

$$\psi(2,a) = \frac{1}{2}\sum_{k=1}^{a-1}\frac{(-1)^{k-1}}{(a-1)_k} - \frac{(-1)^a e E_1(1)}{2\Gamma(a)}$$

Where $E_1(x)$ is the generalised exponential integral.

Proposed by Angad Singh-Pune-India

Solution by proposer

Observe that,

$$\psi(2,a) = \int_0^\infty \frac{e^{-x^2}x}{(1+x^2)^a}dx \overset{1+x^2=t}{=} \frac{1}{2}\int_1^\infty \frac{e^{-(t-1)}}{t^a}dt = \frac{e}{2}\int_1^\infty \frac{e^{-t}}{t^a}dt$$

We know that, generalised exponential integral is defined as,

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}dt$$

Thus, $\psi(2,a) = \frac{e}{3}E_a(1)$

Now, if we integrate $E_a(1)$ by parts, we obtain the following relation,

$$E_a(1) = \frac{e^{-1}}{a-1} - \frac{1}{a-1}E_{a-1}(1)$$

Thus,

$$E_a(1) = e^{-1}\sum_{k=1}^{a-1}\frac{(-1)^{k-1}}{(a-1)_k} - \frac{(-1)^a e E_1(1)}{\Gamma(a)}$$

Finally,

$$\psi(2,a) = \frac{e}{2}E_a(1) = \frac{1}{2}\sum_{k=1}^{a-1}\frac{(-1)^{k-1}}{(a-1)_k} - \frac{(-1)^a e E_1(1)}{2\Gamma(a)}$$

Note by editor:
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