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For $n > 0$, let $U(n) = \int_0^{\infty} (1 - x\sin x)\log(e^{-nx} + 1)dx$

Prove that:

$$\int_1^{\infty} \frac{U(n)}{n^2} dn = \frac{\pi^2}{24} + \frac{1}{2} + \log 2 - \log\pi - \frac{\pi}{2\sinh\pi} + \log\left(\tanh\frac{\pi}{2}\right)$$

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Solution by Rana Ranino-Setif-Algerie

Before doing the job let's warm up with some beautiful series using Weierstrass Product

$$\frac{\sinh x}{x} = \prod_{k=1}^{\infty}\left(1 + \frac{x^2}{k^2x^2}\right) \overset{x=\frac{\pi}{2}}{\Longrightarrow} \frac{2}{\pi}\sinh\frac{\pi}{2} = \prod_{k=1}^{\infty}\left(1 + \frac{1}{4k^2}\right)$$

$$S_1 = \sum_{k=1}^{\infty} \log\left(1 + \frac{1}{4k^2}\right) = \log 2 - \log\pi + \log\left(\sinh\frac{\pi}{2}\right)$$

$$\frac{\sinh x}{x} = \prod_{k=1}^{\infty}\left(1 + \frac{x^2}{k^2x^2}\right) \Rightarrow \log(\sinh x) - \log x = \sum_{k=1}^{\infty} \log\left(1 + \frac{1}{k^2\pi^2}\right)$$

Derivate both side, we get:

$$\sum_{k=1}^{\infty} \frac{2x}{k^2\pi^2 + x^2} = \frac{\pi}{4}\coth\frac{\pi}{2} - \frac{1}{2}$$

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$$coshx=\prod_{k=1}^{\infty}\left(1+\frac{4x^2}{(2k-1)^2\pi^2}\right)\Rightarrow log(coshx)=\sum_{k=1}^{\infty}log\left(1+\frac{4k^2}{(2k-1)^2\pi^2}\right),x=\frac{\pi}{2}$$

$$S_3=\sum_{k=1}^{\infty}log\left(1+\frac{1}{(2k-1)^2}\right)=log\left(cosh\frac{\pi}{2}\right)$$

$$coshx=\prod_{k=1}^{\infty}\left(1+\frac{4x^2}{(2k-1)^2\pi^2}\right)\Rightarrow log(coshx)=\sum_{k=1}^{\infty}log\left(1+\frac{4x^2}{(2k-1)^2\pi^2}\right)$$

Derivate both side, we get:

$$tanhx=\sum_{k=1}^{\infty}\frac{8x}{4x^2+(2k-1)^2\pi^2}\overset{x=\frac{\pi}{2}}{\Longrightarrow}tanh\frac{\pi}{2}=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{1+(2k-1)^2}$$

$$S_4=\sum_{k=1}^{\infty}\frac{1}{1+(2k-1)^2}=\frac{\pi}{4}tanh\frac{\pi}{2}$$

$$U(n)=\int_0^{\infty}(1-xsin\,x)log(e^{-nx}+1)dx=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\infty}(1-xsinx)e^{-nkx}\,dx=$$

$$=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\infty}e^{-nkx}dx+\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\infty}xsinx\,e^{-nkx}dx=$$

$$=-\frac{1}{n}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k}L_x\{xsinx\}_{s=nk}$$

$$U(n)=\frac{\pi^2}{12n}+\sum_{k=1}^{\infty}\frac{(-1)^k2n}{(1+n^2k^2)^2}$$

$$\Omega=\int_1^{\infty}\frac{U(n)}{n^2}dn=\int_0^1U\left(\frac{1}{n}\right)dn\,;U\left(\frac{1}{n}\right)=\frac{\pi^2}{12}n+2\sum_{k=1}^{n}\frac{(-1)^kn^3}{(n^2+k^2)^2}$$

$$\Omega=\int_0^1\left(\frac{\pi^2}{12}n+2\sum_{k=1}^{n}\frac{(-1)^kn^3}{(n^2+k^2)^2}\right)dn=\frac{\pi^2}{24}+\sum_{k=1}^{\infty}(-1)^k\int_0^1\frac{2n^3}{(n^2+k^2)^2}dn$$

$$I=\int_0^1\frac{2n^3}{(n^2+k^2)^2}dn\overset{x=n^2}{=}\int_0^1\frac{x}{(x+k^2)^2}dx=\int_0^1\left(\frac{1}{x+k^2}-\frac{k^2}{(x+k^2)^2}\right)dx=$$

$$=\left[\frac{k^2}{k^2+x}+\log(x+k^2)\right]_0^1=\log\left(1+\frac{1}{k^2}\right)-\frac{1}{1+k^2}$$

$$\Omega=\frac{\pi^2}{24}+\sum_{k=1}^{\infty}(-1)^k\left(\log\left(1+\frac{1}{k^2}\right)-\frac{1}{1+k^2}\right)=\frac{\pi^2}{24}+\sum_{k=1}^{\infty}(-1)^k\log\left(1+\frac{1}{k^2}\right)-$$

$$-\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2}=\frac{\pi^2}{24}+\sum_{k=1}^{\infty}\log\left(1+\frac{1}{4k^2}\right)-\sum_{k=1}^{\infty}\frac{1}{1+4k^2}-$$

$$-\sum_{k=1}^{\infty}\log\left(1+\frac{1}{(2k-1)^2}\right)+\sum_{k=1}^{\infty}\frac{1}{1+(2k-1)^2}=\frac{\pi^2}{24}+S_1-S_2-S_3+S_4$$

$$\Omega=\frac{\pi^2}{24}+\log 2-\log\pi+\log\left(\sinh\frac{\pi}{2}\right)-\log\left(\cosh\frac{\pi}{2}\right)+\frac{1}{2}+\frac{\pi}{4}\tanh\frac{\pi}{2}=$$

$$=\frac{\pi^2}{24}+\frac{1}{2}+\log 2-\log\pi+\log\left(\tanh\frac{\pi}{2}\right)-\frac{\pi}{4}\left[\coth\frac{\pi}{2}-\tanh\frac{\pi}{2}\right]$$

$$\int_1^{\infty}\frac{U(n)}{n^2}dn=\frac{\pi^2}{24}+\frac{1}{2}+\log 2-\log\pi-\frac{\pi}{2\sinh\pi}+\log\left(\tanh\frac{\pi}{2}\right)$$

$$\text{where } U(n)=\int_0^{\infty}(1-x\sin x)\log(e^{-nx}+1)dx\,,n>0$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.