R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

In acute ΔABC the following relationship holds:

$$\cos(A-B)\cos(B-C)\cos(C-A) \le \frac{64abc}{9(a+b)(b+c)(c+a)-8abc}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan
Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\cos(A-B)\cos(B-C)\cos(C-A)$$
$$=\left(2\cos^2\frac{A-B}{2}-1\right)\left(2\cos^2\frac{B-C}{2}-1\right)\left(2\cos^2\frac{C-A}{2}-1\right)$$
$$\overset{(a)}{=} 8\prod\cos^2\frac{B-C}{2}-4\left(\prod\cos^2\frac{B-C}{2}\right)\sum\sec^2\frac{B-C}{2}+2\sum\cos^2\frac{B-C}{2}-1$$

$$\text{Now, }\sum\cos^2\frac{B-C}{2}=\sum\frac{(b+c)^2\sin^2\frac{A}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}=\frac{1}{16R^2s}\sum\frac{bc(b+c)^2}{s-a}$$
$$=\frac{1}{16R^2s}\sum\frac{bc(s+s-a)^2}{s-a}$$
$$=\frac{1}{16R^2s}\sum\left\{\frac{bcs^2}{s-a}+2sbc+bc(s-a)\right\}=\frac{1}{16R^2s}\left\{s^3\sum\sec^2\frac{A}{2}+3s\sum ab-3abc\right\}$$

$$= \frac{1}{16R^2 s}\left[s^3\left\{\frac{s^2 + (4R + r)^2}{s^2}\right\} + 3s(s^2 + 4Rr + r^2) - 12Rrs\right] = \frac{4s^2 + (4R + r)^2 + 3r^2}{16R^2}$$

$$\Rightarrow \sum \cos^2\frac{B - C}{2} \overset{(1)}{=} \frac{4s^2 + (4R + r)^2 + 3r^2}{16R^2}$$

$$\text{Again, } \sum \sec^2\frac{B - C}{2} = \sum \frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{(b + c)^2\sin^2\frac{A}{2}} = \sum \frac{16R^2 s(s - a)a}{4Rrs(b + c)^2}$$

$$= \frac{2R}{r}\sum \frac{a(b + c - a)}{(b + c)^2} \overset{(2)}{=} \frac{2R}{r}\left\{\sum \frac{a}{b + c} - \sum \frac{a^2}{(b + c)^2}\right\}$$

$$\text{Now, } \sum \frac{a}{b + c} = \frac{\sum a(c + a)(a + b)}{\prod(b + c)} = \frac{\sum a(\sum ab + a^2)}{2s(s^2 + 2Rr + r^2)}$$

$$= \frac{2s(s^2 + 4Rr + r^2) + 2s(s^2 - 6Rr - 3r^2)}{2s(s^2 + 2Rr + r^2)} \overset{(3)}{=} \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2}$$

$$\text{and, } \sum \frac{a^2}{(b + c)^2} = \sum \frac{\left(2s - (b + c)\right)^2}{(b + c)^2}$$

$$= \sum \frac{4s^2 - 4s(b + c) + (b + c)^2}{(b + c)^2} \overset{(i)}{=} 4s^2\left[\frac{\sum\{(c + a)^2(a + b)^2\}}{\{\prod(b + c)\}^2}\right] - 4s\left[\frac{\sum(c + a)(a + b)}{\prod(b + c)}\right] + 3$$

$$\sum\{(c + a)^2(a + b)^2\} = \sum\left(\sum ab + a^2\right)^2 = \sum\left\{\left(\sum ab\right)^2 + 2a^2\sum ab + a^4\right\}$$

$$= 3\left(\sum ab\right)^2 + 2\left(\sum ab\right)(\sum a^2) + (\sum a^2)^2 - 2\sum a^2b^2$$

$$= \left(\sum ab\right)^2 + 2\left(\sum ab\right)(\sum a^2) + (\sum a^2)^2 + 2\sum a^2b^2 + 4abc(2s) - 2\sum a^2b^2$$

$$= \left(\sum ab + \sum a^2\right)^2 + 32Rrs^2$$

$$= (3s^2 - 4Rr - r^2)^2 + 32Rrs^2$$

$$\therefore \sum\{(c + a)^2(a + b)^2\} \overset{(ii)}{=} (3s^2 - 4Rr - r^2)^2 + 32Rrs^2$$

$$\text{Again, } \sum(c + a)(a + b) = \sum\left(\sum ab + a^2\right) = 3\sum ab + \sum a^2$$

$$= \sum a^2 + 2\sum ab + \sum ab = 4s^2 + s^2 + 4Rr + r^2$$

$$\therefore \sum(c + a)(a + b) \overset{(iii)}{=} 5s^2 + 4Rr + r^2$$

$$\because \prod(b + c) = s^2 + 2Rr + r^2 \therefore (i), (ii), (iii) \Rightarrow \sum \frac{a^2}{(b + c)^2}$$

$$= \frac{4s^2\{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2\}}{4s^2(s^2 + 2Rr + r^2)^2} - \frac{4s(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} + 3$$

$$= \frac{(3s^2-4Rr-r^2)^2+32Rrs^2-2(5s^2+4Rr+r^2)(s^2+2Rr+r^2)+3(s^2+2Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$= \frac{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}$$

$$\Rightarrow \sum \frac{a^2}{(b+c)^2} \overset{(4)}{=} \frac{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}$$

$$(2),(3),(4) \Rightarrow \sum \sec^2\frac{B-C}{2}$$

$$= \frac{2R}{r}\left\{\frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2} - \frac{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4}{(s^2+2Rr+r^2)^2}\right\}$$

$$\overset{(5)}{=} \frac{2R}{r}\left[\frac{(2s^2-2Rr-2r^2)(s^2+2Rr+r^2)-\{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4\}}{(s^2+2Rr+r^2)^2}\right]$$

$$\text{Also, } 8\prod\cos^2\frac{B-C}{2} = 8\prod\frac{(b+c)^2\sin^2\frac{A}{2}}{a^2}$$

$$= 8\left\{\frac{4s^2(s^2+2Rr+r^2)^2}{16R^2r^2s^2}\right\}\left(\frac{r^2}{16R^2}\right) \overset{(6)}{=} \frac{(s^2+2Rr+r^2)^2}{8R^4}$$

$$(a),(1),(5),(6) \Rightarrow \cos(A-B)\cos(B-C)\cos(C-A) = \frac{(s^2+2Rr+r^2)^2}{8R^4}$$

$$-\left\{\frac{(s^2+2Rr+r^2)^2}{16R^4}\right\}\frac{2R}{r}\left[\frac{(2s^2-2Rr-2r^2)(s^2+2Rr+r^2)-\{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4\}}{(s^2+2Rr+r^2)^2}\right]$$

$$+\frac{4s^2+(4R+r)^2+3r^2}{8R^2}-1$$

$$\Rightarrow \cos(A-B)\cos(B-C)\cos(C-A) \overset{(m)}{=} \frac{r(s^2+2Rr+r^2)^2-R\sigma+R^2r\{4s^2+(4R+r)^2+3r^2\}-8R^4r}{8R^4r}$$

$$(\text{where } \sigma = (2s^2-2Rr-2r^2)(s^2+2Rr+r^2) - \{2s^4-s^2(8Rr+12r^2)+12R^2r^2+8Rr^3+2r^4\})$$

$$\frac{64abc}{9(a+b)(b+c)(c+a)-8abc} \geq \frac{2r}{R} \Leftrightarrow \frac{64.4Rrs}{9.2s.(s^2+2Rr+r^2)-8.4Rrs} \geq \frac{2r}{R} \Leftrightarrow 64R^2$$

$$\geq 9(s^2+2Rr+r^2)-16Rr$$

$$\Leftrightarrow 9s^2 \overset{(7)}{\leq} 64R^2-2Rr-9r^2$$

$$\text{Now, } 9s^2 \overset{\text{Gerretsen}}{\leq} 9(4R^2+4Rr+3r^2) \overset{?}{\leq} 64R^2-2Rr-9r^2$$

$$\Leftrightarrow 14R^2-19Rr-18r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(14R+9r) \overset{?}{\geq} 0 \rightarrow \text{true (Euler)}$$

$$\therefore \frac{64abc}{9(a+b)(b+c)(c+a)-8abc} \overset{(n)}{\geq} \frac{2r}{R}$$

$\therefore$ (m), (n) $\Rightarrow$ it suffices to prove

$$: \frac{r(s^2+2Rr+r^2)^2 - R\sigma + R^2r\{4s^2+(4R+r)^2+3r^2\} - 8R^4r}{8R^4r} - \frac{2r}{R} \leq 0$$

$$\Leftrightarrow \frac{r(s^2+2Rr+r^2)^2 - R\sigma + R^2r\{4s^2+(4R+r)^2+3r^2\} - 8R^4r - 16R^3r^2}{8R^4r} \leq 0$$

$$\Leftrightarrow s^4 + 8R^4 - s^2(6R^2+8Rr-2r^2) + 8R^3r + 22R^2r^2 + 8Rr^3 + r^4 \overset{(u)}{\leq} 0$$

$\because \Delta$ ABC is acute – angled, Walker and Gerretsen

$$\Rightarrow (s^2-2R^2-8Rr-3r^2)(s^2-4R^2-4Rr-3r^2) \leq 0$$

$\Rightarrow$ in order to prove (u),

it suffices to prove : $s^4 + 8R^4 - s^2(6R^2+8Rr-2r^2) + 8R^3r + 22R^2r^2 + 8Rr^3 + r^4$

$$\leq (s^2-2R^2-8Rr-3r^2)(s^2-4R^2-4Rr-3r^2)$$

$$\Leftrightarrow (R+2r)s^2 \overset{(v)}{\leq} 8R^3 + 7R^2r + 7Rr^2 + 2r^3$$

Now, $(R+2r)s^2 \overset{\text{Gerretsen}}{\leq} (R+2r)(4R^2+4Rr+3r^2) \overset{?}{\leq} 8R^3 + 7R^2r + 7Rr^2 + 2r^3$

$$\Leftrightarrow 4t^3 - 5t^2 - 4t - 4 \overset{?}{\geq} 0 \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t^2+3t+2) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (v) \Rightarrow (u) \text{ is true}$$

$$\therefore \cos(A-B)\cos(B-C)\cos(C-A) \leq \frac{64abc}{9(a+b)(b+c)(c+a)-8abc} \ (Proved)$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

In any triangle ABC: $cos(B-C) \leq \frac{h_a}{m_a}$; $(and\ analogs)$

$$m_am_bm_c \geq r_ar_br_c (*)$$

$$cos(A-B)cos(B-C)cos(C-A) \leq \frac{h_ah_bh_c}{m_am_bm_c} \overset{(*)}{\leq} \frac{h_ah_bh_c}{r_ar_br_c} = \frac{2r}{R}$$

$$\frac{64abc}{9(a+b)(b+c)(c+a)-8abc}=\frac{64\cdot 4Rr}{9(s^2+2Rr+r^2)-16Rr}=\frac{64\cdot 2Rr}{9s^2+2Rr+9r^2}$$

We need to prove:

$$\frac{2r}{R}\le\frac{64\cdot 2Rr}{9s^2+2Rr+9r^2}\Leftrightarrow 9s^2+2Rr+9r^2\le 64R^2\Leftrightarrow$$

$$9s^2\le 64R^2-2Rr-9r^2;\ \ (1)$$

But from $s^2\le 4R^2+4Rr+3r^2$ **we have:**

$$9s^2\le 36R^2+36Rr+27r^2\overset{(2)}{\le}64R^2-2Rr-9r^2$$

$$(2)\Leftrightarrow 28R^2-38Rr-36r^2\ge 0\Leftrightarrow 14R^2-19Rr-18r^2\ge 0\Leftrightarrow$$

$$(R-2r)(14R+9r)\ge 0 \text{ true by } R\ge 2r\ (Euler)$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.