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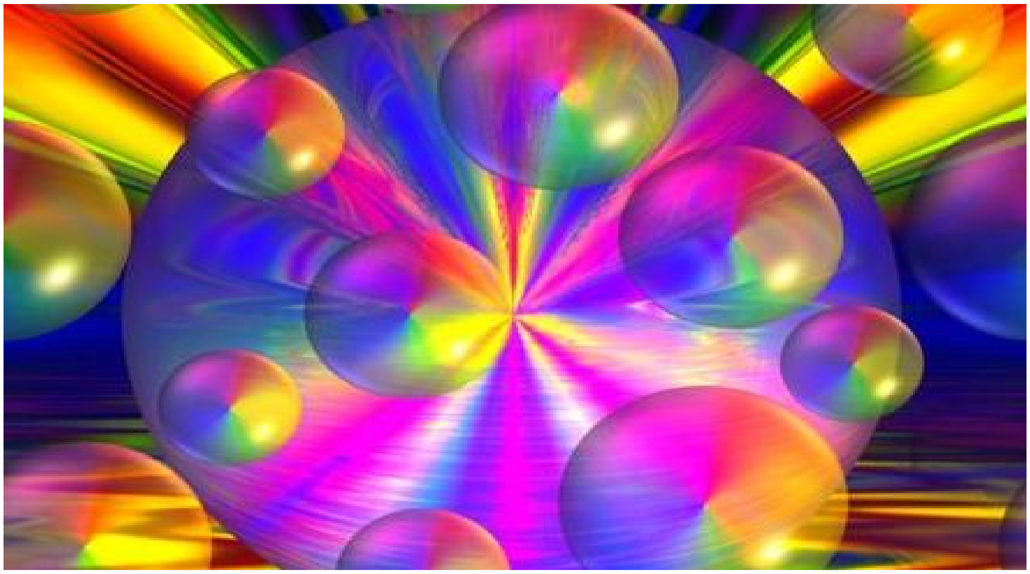

In any ΔABC the following relationship holds:

$$\sqrt[4]{\frac{m_a m_b m_c w_a w_b w_c}{r^6}} \ge \sum\left(\frac{n_a}{h_a} + \frac{2r_a}{s + n_a}\right)$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c)$$

$$\Rightarrow s(b^2 + c^2) - bc(2s - a) = an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc = an_a^2 + a(as - s^2)$$

$$\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc)$$

$$= as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)}$$

$$= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s - a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 + 2h_a r_a = s^2$$

$$\Rightarrow \frac{n_a}{h_a} + \frac{2r_a}{s + n_a} = \frac{sn_a + n_a^2 + 2h_a r_a}{h_a(s + n_a)} = \frac{sn_a + s^2}{h_a(s + n_a)}$$

$$= \frac{s(s + n_a)}{h_a(s + n_a)} \Rightarrow \frac{n_a}{h_a} + \frac{2r_a}{s + n_a} = \frac{s}{h_a} \text{ and analogs} \overset{\text{summing up}}{\Rightarrow} \sum\left(\frac{n_a}{h_a} + \frac{2r_a}{s + n_a}\right) = s\sum\frac{1}{h_a}$$

$$\Rightarrow \sum\left(\frac{n_a}{h_a} + \frac{2r_a}{s + n_a}\right) \overset{(1)}{=} \frac{s}{r}$$

$$m_a w_a \overset{Ioscu}{\geq} \left(\frac{b+c}{2}\cos\frac{A}{2}\right)\frac{2bc\cos\frac{A}{2}}{b+c} = bc\frac{s(s-a)}{bc} \Rightarrow m_a w_a \geq s(s-a) \text{ and analogs}$$

$$\Rightarrow \prod (m_a w_a) \geq \prod \big(s(s-a)\big) = s^4 r^2$$

$$\Rightarrow \sqrt[4]{\frac{m_a m_b m_c w_a w_b w_c}{r^6}} \geq \sqrt[4]{\frac{s^4 r^2}{r^6}} = \frac{s}{r} \overset{\text{by }(1)}{\cong} \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a}\right) \ (\textit{Proved})$$