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In ΔABC the following relationship holds:

$$S \le \frac{abc(b+c)w_a}{(a+b)(a+c)(b+c-a)}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$b+c-a = 4R\cos\frac{A}{2}\cos\frac{B-C}{2} - 4R\cos\frac{A}{2}\sin\frac{A}{2} = 4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right)$$

$$= 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow b+c-a \stackrel{(1)}{=} 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\textit{Now},\ S(a+b)(a+c)(b+c-a) \stackrel{\text{by }(1)}{=} S\left(4R\cos\frac{C}{2}\cos\frac{A-B}{2}\right)\left(4R\cos\frac{B}{2}\cos\frac{A-C}{2}\right)8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\le 2rs(4R)^3\cos\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\left(\because 0 < \cos\frac{A-B}{2}, \cos\frac{A-C}{2} \le 1\right)$$

$$= 2\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)(4R)^3\cos\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 2\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(4R\sin\frac{B}{2}\cos\frac{B}{2}\right)^2\left(4R\sin\frac{C}{2}\cos\frac{C}{2}\right)^2\cos\frac{A}{2}$$

$$= 2ab^2c^2\left(\frac{\cos\frac{A}{2}}{b+c}\right)(b+c) = abc(b+c)\left(\frac{2bc\cos\frac{A}{2}}{b+c}\right) = abc(b+c)w_a$$

$$\Rightarrow S \le \frac{abc(b+c)w_a}{(a+b)(a+c)(b+c-a)}\ (\mathit{Proved})$$