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In ΔABC the following relationship holds:

$$2+\sum_{cyc}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right)=\frac{4R}{r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Daniel Sitaru-Romania, Solution 2,3 by Alex Szoros-Romania, Solution 4 by Mansur Mansurov-Azerbaijan

Solution 1 by Daniel Sitaru-Romania

$$2+\sum_{cyc}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right)=1+\sum_{cyc}\frac{r_a^2+r_b^2}{r_ar_b}=2+\frac{1}{r_ar_br_c}\sum_{cyc}r_c\left(r_a^2+r_b^2\right)=$$

$$=2+\frac{1}{r_ar_br_c}\left((r_a+r_b+r_c)(r_ar_b+r_br_c+r_cr_a)-3r_ar_br_c\right)=$$

$$=2+\frac{(r_a+r_b+r_c)(r_ar_b+r_br_c+r_cr_a)}{r_ar_br_c}-3=$$

$$=\frac{(4R+r)s^2}{rs^2}-1=\frac{4R+r}{r}-1=\frac{4R}{r}+1-1=\frac{4R}{r}$$

Solution 2 by Alex Szoros-Romania

Denote: $s-a=x, s-b=y, s-c=z\Rightarrow x+y+z=s, a=y+z, b=x+z, c=x+y$

$$LHS = 2+\sum_{cyc}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right) = 2+\sum_{cyc}\left(\frac{s-b}{s-a}+\frac{s-c}{s-a}\right) = 2+\sum_{cyc}\frac{a}{s-a} =$$

$$= 2+\sum_{cyc}\frac{y+z}{x} = 2+\frac{\sum yz(y+z)}{xyz}; (1)$$

Using identities: $\frac{abc}{4F} = R, \frac{F}{s} = r$, we have:

$$\frac{4R}{r} = \frac{4abc}{4F}\frac{s}{F} = \frac{abc\,s}{F^2}$$

$$RHD = \frac{4R}{r} = \frac{abc}{(s-a)(s-b)(s-c)} = \frac{(x+y)(y+z)(z+x)}{xyz} =$$

$$= \frac{2xyz+\sum xy(x+y)}{xyz} = 2+\frac{\sum xy(x+y)}{xyz}; (2)$$

From $(1)\&(2)$, we get:

$$2+\sum_{cyc}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right) = \frac{4R}{r}$$

Solution 3 by Alex Szoros-Romania

$$\left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{r_a}\right) = 3+\sum_{cyc}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right); (1)$$

$$\left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{r_a}\right) = (4R+r)\frac{1}{r} = \frac{4R}{r}+1; (2)$$

From $(1)\&(2)$, we get:

$$3+\sum_{cyc}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right) = \frac{4R}{r}+1$$

Therefore,

$$2+\sum_{cyc}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right) = \frac{4R}{r}$$

Solution 4 by Mansur Mansurov-Azerbaijan

$$2+\sum_{cyc}\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right) = \frac{r_a}{r_b}+1+\frac{r_b}{r_a}+1+\frac{r_a+r_b}{r_c}+r_c\frac{r_a+r_b}{r_ar_b} =$$

$$= \frac{r_a + r_b}{r_b} + \frac{r_a + r_b}{r_a} + \frac{r_a + r_b}{r_c} + (r_a + r_b)\frac{r_c}{r_a r_b} =$$

$$= (r_a + r_b)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} + \frac{r_c}{r_a r_b}\right) = (r_a + r_b)\left(\frac{1}{r} + \frac{r_c}{r_a r_b}\right) =$$

$$= F\left(\frac{1}{s-a} + \frac{1}{s-b}\right)\left(\frac{s}{F} + \frac{(s-a)(s-b)}{F(s-c)}\right) =$$

$$= \frac{c}{(s-a)(s-b)}\left(\frac{s(s-c) + (s-a)(s-b)}{s-c}\right) =$$

$$= \frac{c(2s^2 - s(a+b+c) + ab)}{(s-a)(s-b)(s-c)} = \frac{abc}{(s-a)(s-b)(s-c)} =$$

$$= \frac{abc}{\frac{F^2}{s}} = \frac{abc}{rF} = \frac{abc}{r\frac{abc}{4R}} = \frac{4R}{r}$$

Therefore,

$$2 + \sum_{cyc}\left(\frac{r_a}{r_b} + \frac{r_b}{r_a}\right) = \frac{4R}{r}$$

Note by editor:
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