R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Let $b_1, b_2, \dots, b_n \in \mathbb{R}$ such that for each $n \geq 1$, $b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3}$.
If N be the least positive integer satisfying the inequality:

$$\sum_{n=1}^{N} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} \geq \frac{2021}{1015}$$

Find the value of N.

Proposed by Rajeev Rastogi-India

Solution by Adrian Popa-Romania

$$b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3} \overset{Bergstrom}{\geq} \frac{(b_1 + b_2 + \cdots + b_n)^2}{1^3 + 2^3 + \cdots + n^3}$$

$$\Rightarrow \left(\frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n}\right)^2 \geq \frac{1}{\left(\frac{n(n+1)}{2}\right)^2} \Rightarrow \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} \geq \frac{2}{n(n+1)}$$

$$\sum_{n=1}^{N} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} \geq \sum_{n=1}^{N} \frac{2}{n(n+1)} = 2\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N} - \frac{1}{N+1}\right)$$

$$\Rightarrow 9N \geq 2021 \Rightarrow N \geq \frac{2021}{9} = 224,(5), N \in \mathbb{N} \Rightarrow N = 225.$$

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.