On the values of $\tau(6N + 5)$ and $\tau(6N + 7)$ and equivalent statement related to infinitely many sixy prime duplets, triplets and quadraples

Safal Das Biswas

Indian Statistical Institute, Bangalore India , 1st Year Student

Abstract: In this paper we will deal with divisor function in number theory and some of the special values $\tau(6N + 5)$ and $\tau(6N + 7)$ and some related statement to sixy primes.

- ## Contents:

1. Introduction

Let, $n > 1$ be an natural number, $\tau(n) = \#\{d|n : d > 0, d \in \mathbb{N}\}$. We know that we have a formule for $\tau(n)$. If $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ is the prime factorization for n, then we have $\tau(n) = \prod_{i=1}^{r} (a_i + 1)$. Thus $\tau(p) = 2$ for p is prime. We will deal with exact values of $\tau(6N + 5)$ and $\tau(6N + 7)$ where N is a natural number, without knowing there prime decompositions. We will go on in the paper to seek by prime progressed by 6 i. e. if p and $p + 6$ are primes then $\tau(p) + \tau(p + 6) = 4$, which will be of interest. Also we will search for the related statement regarding sixy prime conjectures.

2. Propositions regarding $\tau(6N + 5)$ and $\tau(6N + 7)$

Proposition 1.1: $\tau(6N + 5) = \left(2 \sum_{m=1}^{\left\lfloor \frac{N+2}{7} \right\rfloor} \left\lfloor \frac{N + 1 - m}{6m - 1} \right\rfloor - 2 \sum_{m=1}^{\left\lfloor \frac{N+1}{7} \right\rfloor} \left\lfloor \frac{N - m}{6m - 1} \right\rfloor \right) + 2$, for $N \in \mathbb{N}$

Proof:

Define

- $\Omega = \{(m, n) : (6n + 1)(6m - 1) \le 6N - 1, \text{where } (m, n) \in \mathbb{N}^2\}$
- $A_N = \{(m, n) : (6m - 1)(6n + 1) = N, (m, n) \in \mathbb{N}^2\}$ for all $N \in \mathbb{N}$

Observation : $\Omega = \coprod_{m=1}^{N} A_{6m-1}$ where, $\coprod$ denotes the disjoint union

Claim 1: $|\Omega| = \sum_{m=1}^{\left\lfloor \frac{N+1}{7} \right\rfloor} \left\lfloor \frac{N - m}{6m - 1} \right\rfloor$

Proof: For $(6n + 1)(6m - 1) \le 6N - 1$, we have $6n + 1 \le \frac{6N - 1}{6m - 1}$ so, $n \le \frac{N - m}{6m - 1}$ and we need $\frac{N - m}{6m - 1} \ge 1$ so $m \le \left\lfloor \frac{N + 1}{7} \right\rfloor$ for each $m \le \left\lfloor \frac{N + 1}{7} \right\rfloor$ we get some values n, such that

$(6n+1)(6m-1) \leq 6N-1$ holds. and there would be $\left\lfloor \frac{N-m}{6m-1} \right\rfloor$ such n's for each m. Thus we have

$$|\Omega| = \sum_{m=1}^{\left\lfloor \frac{N+1}{7} \right\rfloor} \left\lfloor \frac{N-m}{6m-1} \right\rfloor \qquad \blacksquare$$

Claim 2: $\left| \coprod_{m=1}^{N} A_{6m-1} \right| = \sum_{m=1}^{N} \frac{\tau(6m-1)-2}{2}$

Proof: We need to calculate the cardinality of each $A_{i,}$ as $\left| \coprod_{m=1}^{N} A_{6m-1} \right| = \sum_{m=1}^{N} |A_{6m-1}|$

For $|A_{6m-1}|$ we need, The number of ways we can can write $6m-1$ as the product of integers , Suppose $d_1 < d_2, \ldots < d_k$ be the divisors of $6m-1$, excluding 1 and $6m-1$

$6m-1 = d_1 d_k = d_2 d_{k-1} = \cdots$, Note that there are $\tau(6m-1)$ divisors in total , including 1 and $6m-1$, excluding those 2 we have $\tau(6m-1)-2$ divisors, $6m-1$ can be witten as product of two distnct divisors , if divisor exist one divisor is form $6k-1$ and other is of form $6r+1$, and $6m-1$ can be written as the product of two divisor in $\frac{\tau(6m-1)-2}{2}$ ways thus $|A_{6m-1}| = \frac{\tau(6m-1)-2}{2}$

$$\left| \coprod_{m=1}^{N} A_{6m-1} \right| = \sum_{m=1}^{N} |A_{6m-1}| = \sum_{m=1}^{N} \frac{\tau(6m-1)-2}{2} \qquad \blacksquare$$

By combining Claim 1 and Claim 2

As $\Omega = \coprod_{m=1}^{N} A_{6m-1}$, we have, $\sum_{m=1}^{\left\lfloor \frac{N+1}{7} \right\rfloor} \left\lfloor \frac{N-m}{6m-1} \right\rfloor = |\Omega| = \left| \coprod_{m=1}^{N} A_{6m-1} \right| = \sum_{m=1}^{N} \frac{\tau(6m-1)-2}{2}$

thus rearranging gives, $\sum_{m=1}^{N} \tau(6m-1) = 2\left(\sum_{m=1}^{\left\lfloor \frac{N+1}{7} \right\rfloor} \left\lfloor \frac{N-m}{6m-1} \right\rfloor \right) + 2N$

Thus, $\tau(6N+5) = \sum_{m=1}^{N+1} \tau(6m-1) - \sum_{m=1}^{N} \tau(6m-1) = \left(2 \sum_{m=1}^{\left\lfloor \frac{N+2}{7} \right\rfloor} \left\lfloor \frac{N+1-m}{6m-1} \right\rfloor - 2 \sum_{m=1}^{\left\lfloor \frac{N+1}{7} \right\rfloor} \left\lfloor \frac{N-m}{6m-1} \right\rfloor \right) + 2$

and we are done. $\blacksquare$

Proposition 1.2:

$$\tau(6N+7)=\left(\sum_{m=1}^{\left\lfloor\frac{N}{7}\right\rfloor}\left\lfloor\frac{N+1-m}{6m+1}\right\rfloor+\sum_{m=1}^{\left\lfloor\frac{N+2}{5}\right\rfloor}\left\lfloor\frac{N+1+m}{6m-1}\right\rfloor-\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor-\sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor\right)+2$$

Proof:

Define

- $\Omega_N = R_N \sqcup S_N$
  $R_N = \{(m,n,1) : (6m+1)(6n+1) \leq 6N+1 \, , m,n \in \mathbb{N}\}$
  $S_N = \{(m,n,2) : (6m-1)(6n-1) \leq 6N+1 \, , m,n \in \mathbb{N}\}$
  $R_N(j) = \{(m,n,1) : (6m+1)(6n+1) = j \, , m,n \in \mathbb{N}\}$
  $S_N(j) = \{(m,n,2) : (6m-1)(6n-1) = j, m,n \in \mathbb{N}\}$

- Observation :

$$R_N = \coprod_{k=1}^{N} R_N(6k+1)$$

$$S_N = \coprod_{k=1}^{N} S_N(6k+1)$$

$$\Omega_N = R_N \sqcup S_N = \coprod_{k=1}^{N} R_N(6k+1) \sqcup S_N(6k+1)$$

Claim 1. $|\Omega_N| = \displaystyle\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor+\sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor$

Proof: We will start with $|\Omega_N| = |R_N \sqcup S_N| = |R_N| + |S_N|$ now, $|R_N| = \displaystyle\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor$ and

$|S_N| = \displaystyle\sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor$ by the same idea we used in previous proposition claims ■

Claim 2. $|R_N(6k+1) \sqcup S_N(6k+1)| = \tau(6k+1) - 2$

Proof: $|R_N(6k+1) \sqcup S_N(6k+1)|$ is acutally estimating the number of divisors $6k+1$ as if one divisor is congurant 1 mod 6 other must be conguent 1 mod 6 and same for -1 mod 6 as we are excluding 1 and the integer itself so,

$|R_N(6k+1) \sqcup S_N(6k+1)| = \tau(6k+1) - 2$ and we are done. ■

$$|\Omega_N| = |R_N \sqcup S_N| = \left|\coprod_{k=1}^{N} R_N(6k+1) \sqcup S_N(6k+1)\right| = \sum_{k=1}^{N} |R_N(6k+1) \sqcup S_N(6k+1)|$$

$$\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor + \sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor = \sum_{k=1}^{N} |R_N(6k+1) \sqcup S_N(6k+1)| = \sum_{k=1}^{N}(\tau(6k+1) - 2)$$

$$\sum_{k=1}^{N}\tau(6k+1) = \sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor + \sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor + 2N$$

Thus $\tau(6N+7) = 2 + \left(\sum_{m=1}^{\left\lfloor\frac{N}{7}\right\rfloor}\left\lfloor\frac{N+1-m}{6m+1}\right\rfloor + \sum_{m=1}^{\left\lfloor\frac{N+2}{5}\right\rfloor}\left\lfloor\frac{N+1+m}{6m-1}\right\rfloor - \sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor - \sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor\right)$ ■

3. Related Statement to sixy primes, duplets, triplets and quadraples.

Theorem 1. $6N+5$ and $6N+11$ are sixy primes for infininitely many integers N if and only if $\left\lfloor\frac{N+3}{7}\right\rfloor = \left\lfloor\frac{N+1}{7}\right\rfloor$ and $\left\lfloor\frac{N+2-m}{6m-1}\right\rfloor = \left\lfloor\frac{N-m}{6m-1}\right\rfloor$ holds for $m = 1,2,3,\ldots,\left\lfloor\frac{N+1}{7}\right\rfloor$ for all such N.

Proof:

Only if Part, Suppose $6N+5$ and $6N+11$ is prime for infinitley many integers N then

Fix one such N. Define $F(N) = \sum_{m=1}^{N}\tau(6m-1)$ then $F(N+2) - F(N) = 4 =$

$$\tau(6N+5) + \tau(6N+11) = 2\left(\sum_{m=1}^{\left\lfloor\frac{N+3}{7}\right\rfloor}\left\lfloor\frac{N+2-m}{6m-1}\right\rfloor - \sum_{m=1}^{\left\lfloor\frac{N+1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m-1}\right\rfloor\right) + 4 \text{ thus}$$

$\sum_{m=1}^{\left\lfloor\frac{N+3}{7}\right\rfloor}\left\lfloor\frac{N+2-m}{6m-1}\right\rfloor - \sum_{m=1}^{\left\lfloor\frac{N+1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m-1}\right\rfloor = 0$, If $N = 7k-2$ or $7k-3$ then $\left\lfloor\frac{N+3}{7}\right\rfloor \neq \left\lfloor\frac{N+1}{7}\right\rfloor$

but then $6(7k-2)+5 = 42k-7$ thus $7|6N+5$ thus $6N+5$ is not prime, and if

$6(7k-3)+11 = 42k-7$ thus $7|6N+11$ then $6N+11$ is not prime thus $N \neq 7k-2$ or $7k-3$

Thus $\left\lfloor\frac{N+3}{7}\right\rfloor = \left\lfloor\frac{N+1}{7}\right\rfloor = K(\text{say})$ then $S = \sum_{m=1}^{K}\left(\left\lfloor\frac{N+2-m}{6m-1}\right\rfloor - \left\lfloor\frac{N-m}{6m-1}\right\rfloor\right) = 0$,

We know that $\left\lfloor\frac{N+2-m}{6m-1}\right\rfloor \geq \left\lfloor\frac{N-m}{6m-1}\right\rfloor$ thus $S \geq 0$ but $S = 0$ so $\left\lfloor\frac{N+2-m}{6m-1}\right\rfloor = \left\lfloor\frac{N-m}{6m-1}\right\rfloor$,

for $N = 1,2,\dots,K$

If part : $\left\lfloor\frac{N+3}{7}\right\rfloor = \left\lfloor\frac{N+1}{7}\right\rfloor$ and $\left\lfloor\frac{N+2-m}{6m-1}\right\rfloor = \left\lfloor\frac{N-m}{6m-1}\right\rfloor$,

for $N = 1,2,\dots,K$ we have $\sum_{m=1}^{\left\lfloor\frac{N+3}{7}\right\rfloor}\left\lfloor\frac{N+2-m}{6m-1}\right\rfloor - \sum_{m=1}^{\left\lfloor\frac{N+1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m-1}\right\rfloor = 0$ thus

$\tau(6N+5) + \tau(6N+11) = 4$ implying $6N+5$ and $6N+11$ is prime.

Theorem 2. $6N+7$ and $6N+13$ are sixy primes for infininitely many integers N if and only $\left\lfloor\frac{N-1}{7}\right\rfloor = \left\lfloor\frac{N+1}{7}\right\rfloor$ and $\left\lfloor\frac{N+3}{5}\right\rfloor = \left\lfloor\frac{N+1}{5}\right\rfloor$ and $\left\lfloor\frac{N+2-m}{6m+1}\right\rfloor = \left\lfloor\frac{N-m}{6m+1}\right\rfloor$ for all $m = 1,2,\dots,\left\lfloor\frac{N-1}{7}\right\rfloor$ and $\left\lfloor\frac{N+2+m}{6m-1}\right\rfloor = \left\lfloor\frac{N+m}{6m-1}\right\rfloor$ for $m = 1,2,\dots,\left\lfloor\frac{N+1}{5}\right\rfloor$

Proof: Define $F(N) = \left(\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor + \sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor\right) + 2N$ then we have

$4 = \tau(6N+7) + \tau(6N+13) = F(N+2) - F(N) =$

$$\left(\sum_{m=1}^{\left\lfloor\frac{N+1}{7}\right\rfloor}\left\lfloor\frac{N+2-m}{6m+1}\right\rfloor + \sum_{m=1}^{\left\lfloor\frac{N+3}{5}\right\rfloor}\left\lfloor\frac{N+2+m}{6m-1}\right\rfloor - \sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor + \sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor\right) + 4$$

Only if part: Suppose $6N+7$ and $6N+13$ is prime then $N \neq 7k-1$ or $7k$ or $5k-3$ or $5k-2$

Unless $6N+7$ or $6n+13$ is divisible by 7 or $6N+7$ or $6N+13$ is divisible by 5

thus using the same logic as theorem 1 we deduced that $\left\lfloor\frac{N-1}{7}\right\rfloor = \left\lfloor\frac{N+1}{7}\right\rfloor$ and $\left\lfloor\frac{N+3}{5}\right\rfloor = \left\lfloor\frac{N+1}{5}\right\rfloor$ and $\left\lfloor\frac{N+2-m}{6m+1}\right\rfloor = \left\lfloor\frac{N-m}{6m+1}\right\rfloor$ for all $m = 1,2,\dots,\left\lfloor\frac{N-1}{7}\right\rfloor$ and $\left\lfloor\frac{N+2+m}{6m-1}\right\rfloor = \left\lfloor\frac{N+m}{6m-1}\right\rfloor$ for $m = 1,2,\dots,\left\lfloor\frac{N+1}{5}\right\rfloor$

If part: If $\left\lfloor\frac{N-1}{7}\right\rfloor = \left\lfloor\frac{N+1}{7}\right\rfloor$ and $\left\lfloor\frac{N+3}{5}\right\rfloor = \left\lfloor\frac{N+1}{5}\right\rfloor$

and $\left\lfloor\frac{N+2-m}{6m+1}\right\rfloor = \left\lfloor\frac{N-m}{6m+1}\right\rfloor$ for all $m = 1,2,\dots,\left\lfloor\frac{N-1}{7}\right\rfloor$ and $\left\lfloor\frac{N+2+m}{6m-1}\right\rfloor = \left\lfloor\frac{N+m}{6m-1}\right\rfloor$ for

$m = 1,2,\dots,\left\lfloor\frac{N+1}{5}\right\rfloor$ then $\tau(6N+7)+\tau(6N+13) = F(N+2) - F(N) = 4$ ■

Theorem 3. $6N+5, 6N+11, 6N+17$ are sixy primes for infininitely many integers N if and only if $\left\lfloor\frac{N+4}{7}\right\rfloor = \left\lfloor\frac{N+1}{7}\right\rfloor$ and $\left\lfloor\frac{N+3-m}{6m-1}\right\rfloor = \left\lfloor\frac{N-m}{6m-1}\right\rfloor$ holds for $m = 1,2,3,\dots,\left\lfloor\frac{N+1}{7}\right\rfloor$ for all such N.

Proof: Similar idea as Theorem 1.

Theorem 4. $6N+7, 6N+13, 6N+19$ are sixy primes for infininitely many integers N if and only

$\left\lfloor\frac{N-1}{7}\right\rfloor = \left\lfloor\frac{N+2}{7}\right\rfloor$ and $\left\lfloor\frac{N+4}{5}\right\rfloor = \left\lfloor\frac{N+1}{5}\right\rfloor$ and $\left\lfloor\frac{N+3-m}{6m+1}\right\rfloor = \left\lfloor\frac{N-m}{6m+1}\right\rfloor$ for all

$m = 1,2,\dots,\left\lfloor\frac{N-1}{7}\right\rfloor$ and $\left\lfloor\frac{N+3+m}{6m-1}\right\rfloor = \left\lfloor\frac{N+m}{6m-1}\right\rfloor$ for $m = 1,2,\dots,\left\lfloor\frac{N+1}{5}\right\rfloor$

Proof: Similar Idea as Theorem 2

Theorem 5. $6N+5, 6N+11, 6N+17, 6N+23$ are sixy primes for infininitely many integers N if and only if $\left\lfloor\frac{N+5}{7}\right\rfloor = \left\lfloor\frac{N+1}{7}\right\rfloor$ and $\left\lfloor\frac{N+4-m}{6m-1}\right\rfloor = \left\lfloor\frac{N-m}{6m-1}\right\rfloor$ holds for $m = 1,2,3,\dots,\left\lfloor\frac{N+1}{7}\right\rfloor$ for all such N.

Proof: Similar idea as Theorem 1.

Theorem 6. $6N+7, 6N+13, 6N+19, 6N+25$ are sixy primes for infininitely many integers N if and only $\left\lfloor\frac{N-1}{7}\right\rfloor = \left\lfloor\frac{N+3}{7}\right\rfloor$ and $\left\lfloor\frac{N+5}{5}\right\rfloor = \left\lfloor\frac{N+1}{5}\right\rfloor$ and $\left\lfloor\frac{N+3-m}{6m+1}\right\rfloor = \left\lfloor\frac{N-m}{6m+1}\right\rfloor$

$m = 1,2,\dots,\left\lfloor\frac{N-1}{7}\right\rfloor$ and $\left\lfloor\frac{N+3+m}{6m-1}\right\rfloor = \left\lfloor\frac{N+m}{6m-1}\right\rfloor$ for $m = 1,2,\dots,\left\lfloor\frac{N+1}{5}\right\rfloor$

Proof: Similar Idea as Theorem 2

4. Computing the closed form of $F_1(n) = \sum_{m=1}^{n} \tau(6m-1)$ and $F_2(N) = \sum_{m=1}^{N} \tau(6m+1)$

Proposition 2.1:

If, $F_1(n) = \sum_{m=1}^{n} \tau(6m-1)$, then there exist a sequence bounded $\{\nabla_n\}_{n=13(1)\infty}$ such that

$$\frac{1}{e} < \nabla_n < e \text{ for all n, satisfying } F_1(n) = \log\left(\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor}\ \nabla_n e^2\right)^n \quad \text{for } n \geq 13$$

Proof: $F_1(n) = \sum_{m=1}^{n} \tau(6m-1) = 2\sum_{m=1}^{\left\lfloor\frac{n+1}{7}\right\rfloor} \left\lfloor\frac{n-m}{6m-1}\right\rfloor + 2n$, we will use the identity $x - 1 \leq \lfloor x \rfloor < x$

let, $r = \left\lfloor\frac{n+1}{7}\right\rfloor$, then $\sum_{m=1}^{r} \left\lfloor\frac{n-m}{6m-1}\right\rfloor \leq \sum_{i=1}^{r} \frac{n-m}{6m-1} = n\sum_{m=1}^{r} \frac{1}{6m-1} - \frac{1}{6}\sum_{m=1}^{r} \frac{6m}{6m-1} \leq$

$$n\left(\frac{1}{5} + \frac{1}{6}\sum_{m=1}^{r} \frac{1}{m}\right) - \frac{r}{6} \leq n\left(\frac{1}{5} + \frac{1}{6}H_r\right) - \frac{1}{42}(n+1) + \frac{1}{6} \leq n\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{6}\log\left(\left\lfloor\frac{n+1}{7}\right\rfloor\right)\right) - \frac{n}{42} + \frac{1}{7}$$

$$= \frac{12n}{35} + \frac{1}{7} + \frac{n}{6}\log\left(\left\lfloor\frac{n+1}{7}\right\rfloor\right)$$

$$F_1(n) \leq \frac{24n}{35} + \frac{2}{7} + 2n + \frac{n}{3}\log\left(\left\lfloor\frac{n+1}{7}\right\rfloor\right).$$

Now we will find a lower bound for $F_1(n)$

$$\sum_{m=1}^{r} \left\lfloor\frac{n-m}{6m-1}\right\rfloor \geq \sum_{i=1}^{r} \frac{n-m}{6m-1} - r = n\sum_{i=1}^{r} \frac{1}{6m-1} - \frac{1}{6}\sum_{m=1}^{r} \frac{6m}{6m-1} - r \geq \frac{n}{6}H_r - \frac{5}{4}r \geq$$

$$\frac{n}{6}\log\left\lfloor\frac{n+1}{7}\right\rfloor - \frac{5}{4}r \geq \frac{n}{6}\log\left\lfloor\frac{n+1}{7}\right\rfloor - \frac{5}{28}(n+1)$$

Thus, $F_1(n) \geq \frac{n}{3}\log\left\lfloor\frac{n+1}{7}\right\rfloor - \frac{5}{14}n - \frac{5}{14} + 2n$

$$\frac{F_1(n)}{n} \leq \frac{24}{35} + \frac{2}{7n} + 2 + \log\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor} < \frac{26}{35} + \log\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor}\, e^2 < 1 + \log\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor}\, e^2$$

$$\frac{F_1(n)}{n} \geq \log\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor}\, e^2 - \frac{5}{14} - \frac{5}{14n} > -1 + \log\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor}\, e^2$$

Thus $\frac{F_1(n)}{n} = \log\nabla_n + \log\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor}\, e^2$ where $-1 < \log\nabla_n < 1$ thus $\frac{1}{e} < \nabla_n < e$

Hence $F_1(n) = \log\left(\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor}\ \nabla_n e^2\right)^n$

Note that we need the estimate $n \geq 13$ as we need $\frac{n+1}{7} \geq 2$ as we applied estimate of H_r ■

Proposition 2.2:

If, $F_2(N) = \sum_{m=1}^{N} \tau(6m+1)$, then there exist a sequence bounded $\{\Delta_N\}_{N=15(1)\infty}$ such that

$\frac{1}{e} < \Delta_N < e$ for all $N \geq 15$, satisfying $F_2(N) = \log\left(\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor e^2\Delta_N}\right)^N$ for $N \geq 15$.

Proof: $\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor} \left\lfloor\frac{N-m}{6m+1}\right\rfloor \leq \sum_{m=1}^{r} \frac{N-m}{6m+1}$ where, $r = \left\lfloor\frac{N-1}{7}\right\rfloor$ and $N - m > 0$ for all postive integer m

in the range. $\sum_{m=1}^{r} \frac{N-m}{6m+1} \leq \sum_{m=1}^{r} \frac{N-m}{6m} = \frac{N}{6}H_r - \frac{1}{6}r \leq \frac{N}{6}(\log r + 1) - \frac{1}{6}\left(\frac{N-1}{7} - 1\right)$

$$= \frac{N}{6}(\log r + 1) - \frac{N-1}{42} + \frac{1}{6} = \frac{N}{6}\log r + \frac{N}{6} - \frac{N}{42} + \frac{1}{42} + \frac{1}{6} = \frac{N}{6}\log r + \frac{N}{7} + \frac{4}{21}$$

$\sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor} \left\lfloor\frac{N+m}{6m-1}\right\rfloor \leq \sum_{m=1}^{j} \frac{N+m}{6m-1}$ where, $j = \left\lfloor\frac{N+1}{5}\right\rfloor$

$$\sum_{m=1}^{j} \frac{N+m}{6m-1} \leq \frac{N}{5} + \sum_{m=1}^{j-1} \frac{N}{6m+5} + \frac{1}{6}\sum_{m=1}^{j} \frac{6m}{6m-1} \leq \frac{N}{5} + \frac{N}{6}H_j + \frac{1}{4}j \leq \frac{N}{5} + \frac{N}{6}(\log j + 1) + \frac{N+1}{20}$$

$$= \frac{N}{6}\log j + N\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{20}\right) + \frac{1}{20}$$

$$F_2(N) = \sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor} \left\lfloor\frac{N-m}{6m+1}\right\rfloor + \sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor} \left\lfloor\frac{N+m}{6m-1}\right\rfloor + 2N \leq \frac{N}{6}(\log rj) + \frac{215N}{84} + \frac{101}{420}$$

$$= \frac{N}{6}\left(\log\left(\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor\right)\right) + \frac{215N}{84} + \frac{101}{420}$$

$$\frac{F_2(N)}{N} \leq \frac{1}{6}\log\left(\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor\right) + \frac{47}{84} + 2 + \frac{101}{420N} \leq \frac{1}{6}\log\left(\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor\right) + 2 + \frac{4}{5} \leq$$

$$\log\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor e^2} + \log e$$

$$\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor} \left\lfloor\frac{N-m}{6m+1}\right\rfloor \geq \sum_{m=1}^{r} \frac{N-m}{6m+1} - r = \sum_{m=1}^{r} \frac{N}{6m+1} - \frac{1}{6}\sum_{m=1}^{r} \frac{6m}{6m+1} - r$$

$$\sum_{m=1}^{r}\frac{N}{6m+1}=\sum_{m=1}^{r-1}\frac{N}{6m+1}+\frac{N}{6r+1}=\sum_{m=1}^{r-1}\frac{N}{6m+1}+\frac{N}{6\left\lfloor\frac{N-1}{7}\right\rfloor+1}$$

$$6\left\lfloor\frac{N-1}{7}\right\rfloor+1\leq\frac{6}{7}(N-1)+1=\frac{6N+1}{7}\ \Rightarrow\frac{N}{6\left\lfloor\frac{N-1}{7}\right\rfloor+1}\geq\frac{7N}{6N+1}\geq 1$$

$$\sum_{m=1}^{r}\frac{N}{6m+1}\geq\sum_{m=1}^{r-1}\frac{N}{6m+1}+1\geq\frac{N}{6}\left(\sum_{m=0}^{r-1}\frac{1}{m+1}\right)-\frac{N}{6}+1=\frac{N}{6}H_r-\frac{N}{6}+1$$

$$\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor\geq\sum_{m=1}^{r}\frac{N-m}{6m+1}-r=\sum_{m=1}^{r}\frac{N}{6m+1}-\frac{1}{6}\sum_{m=1}^{r}\frac{6m}{6m+1}-r\geq$$

$$\frac{N}{6}H_r-\frac{N}{6}+1-\frac{1}{6}r-r\geq\frac{N}{6}\log r-\frac{N}{6}+1-\frac{7r}{6}\geq\frac{N}{6}\log r-\frac{N}{6}+1-\frac{N-1}{6}=\frac{N}{6}\log r-\frac{N}{3}+\frac{7}{6}$$

$$\sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor\geq\sum_{m=1}^{j}\frac{N+m}{6m-1}-j\geq\sum_{m=1}^{j}\frac{N+m}{6m}-j=\frac{N}{6}H_j+\frac{j}{6}-j\geq\frac{N}{6}\log j-\frac{5j}{6}\geq\frac{N}{6}\log j-\frac{N}{6}-\frac{1}{6}$$

$$F_2(N)=\sum_{m=1}^{\left\lfloor\frac{N-1}{7}\right\rfloor}\left\lfloor\frac{N-m}{6m+1}\right\rfloor+\sum_{m=1}^{\left\lfloor\frac{N+1}{5}\right\rfloor}\left\lfloor\frac{N+m}{6m-1}\right\rfloor+2N\geq\frac{N}{6}\log rj-\frac{N}{2}+2N+1$$

$$\frac{F_2(N)}{N}\geq\frac{1}{6}\log rj-\frac{1}{2}+2+\frac{1}{N}\geq\frac{1}{6}\log rj-\frac{1}{2}+2\geq\log\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor}e^2+\log\frac{1}{e}$$

Thus, $\log\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor}e^2+\log\frac{1}{e}\leq\ \frac{F_2(N)}{N}\leq\log\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor}e^2+\log e$

Hence $\exists$ a sequence call it Δ_N with $\frac{F_2(N)}{N}=\log\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor}e^2+\log\Delta_N$

$F_2(N)=\log\left(\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor}e^2\Delta_N\right)^N$ where, $N\geq 15$ and $\frac{1}{e}<\Delta_N<e$ ■

5. Some remarks(indeed the related statement futhermore to sixy primes)

Remark: 1.6N + 7,6N + 13 is prime if and only if $\log \frac{\left(\sqrt[6]{\left\lfloor\frac{N+1}{7}\right\rfloor\left\lfloor\frac{N+3}{5}\right\rfloor} e^2 \Delta_{N+2}\right)^{N+2}}{\left(\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor} e^2 \Delta_{N}\right)^{N}} = 4$

$$\equiv \frac{\left(\sqrt[6]{\left\lfloor\frac{N+1}{7}\right\rfloor\left\lfloor\frac{N+3}{5}\right\rfloor} e^2 \Delta_{N+2}\right)^{N+2}}{\left(\sqrt[6]{\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor} e^2 \Delta_{N}\right)^{N}} = e^4 \equiv \left(\left\lfloor\frac{N+1}{7}\right\rfloor\left\lfloor\frac{N+3}{5}\right\rfloor\right)^{\frac{N+2}{6}} \Delta_{N+2}^{N+2} = \left(\left\lfloor\frac{N-1}{7}\right\rfloor\left\lfloor\frac{N+1}{5}\right\rfloor\right)^{\frac{N}{6}} \Delta_{N}^{N}$$

2.6n + 5,6n + 11 is prime if and only if $\log \frac{\left(\sqrt[3]{\left\lfloor\frac{n+3}{7}\right\rfloor} \nabla_{n+2} e^2\right)^{n+2}}{\left(\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor} \nabla_{n} e^2\right)^{n}} = 4$

$$\equiv \frac{\left(\sqrt[3]{\left\lfloor\frac{n+3}{7}\right\rfloor} \nabla_{n+2} e^2\right)^{n+2}}{\left(\sqrt[3]{\left\lfloor\frac{n+1}{7}\right\rfloor} \nabla_{n} e^2\right)^{n}} = e^4 \equiv \left(\left\lfloor\frac{n+3}{7}\right\rfloor\right)^{\frac{n+2}{3}} \nabla_{n+2}^{n+2} = \left(\left\lfloor\frac{n+1}{7}\right\rfloor\right)^{\frac{n}{3}} \nabla_{n}^{n}$$

3. Likewise we can do for three consecutive sixy primes and four consecutive sexy primes.

6. Acknowledgement

I am thankful to my Insititute(basically the books from intitute library I mentioned in the references), and my all my undergrad friends for giving helpful suggestions.

7. References.

1. David M Burton：Elementary Number theory.

2. Tom M Apostol：Indroduction to Analytic Number theory.

Indian Statistical Institute, Bangalore India

Email: safaldasbiswas@gmail. com