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If $x, y, z > 0$ then:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)} \geq 10$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Daniel Sitaru-Romania, Solution 2 by Eldeniz Hesenov-Georgia, Solution 3 by Alex Szoros-Romania, Solution 4 by Tran Hong-Dong Thap-Vietnam, Solution 5 by Remus Florin Stanca-Romania, Solution 6 by Dang Le Gia Khanh-Vietnam

Solution 1 by Daniel Sitaru-Romania

$$a=y+z, b=z+x, c=x+y, s=x+y+z, \frac{r}{R}=\frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)}=$$

$$=s\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)+\frac{2r}{R}=$$

$$=s\cdot\frac{4R+r}{rs}+\frac{2r}{R}=\frac{4R}{r}+1+\frac{2r}{R} \geq 10 \leftrightarrow$$

$$4t+\frac{2}{t} \geq 9, t=\frac{R}{r} \geq 2, t \geq 2 \leftrightarrow 4t^2-9t+2 \geq 0 \leftrightarrow$$

$$\leftrightarrow 4t^2-8t-t+2 \geq 0 \leftrightarrow 4t(t-2)-(t-2) \geq 0 \leftrightarrow$$

$$\leftrightarrow (t-2)(4t-1) \geq 0 \leftrightarrow t-2 \geq 0 \ because:$$

$$4t-1 = \frac{4R}{r} - 1 = \frac{(x+y)(y+z)(z+x)}{xyz} - 1 \overset{CESARO}{\geq} \frac{8xyz}{xyz} - 1 = 7 > 0$$

Equality holds for: $a = b = c \leftrightarrow x = y = z$.

Solution 2 by Eldeniz Hesenov-Georgia

In $\Delta ABC: x = s-a, y = s-b, z = s-c$

$$Lhs = s\sum_{cyc}\frac{1}{s-a} + \frac{8\prod(s-a)s}{abcs} =$$

$$= s^2\frac{\sum(s-a)(s-b)}{s\prod(s-a)} + \frac{8F^2}{4SRs} = \frac{r^2+4Rr}{r^2} + \frac{2r}{R} = \frac{r+4R}{r} + \frac{2r}{R} \geq 10 \Leftrightarrow$$

$$\frac{4R}{r} + \frac{2r}{R} \geq 9, t = \frac{R}{r} \geq 2 \Leftrightarrow 4t + \frac{9}{t} \geq 9 \Leftrightarrow (t-2)(4t-1) \geq 0, \forall t \geq 2$$

Solution 3 by Alex Szoros-Romania

$$\left(\sum x\right)\left(\sum \frac{1}{x}\right) = 3 + \sum\frac{x+y}{z}, \forall x,y,z > 0$$

The desired inequality, becomes

$$3 + \sum\frac{x+y}{z} + 8\prod\frac{z}{x+y} \geq 10 \Leftrightarrow \sum\frac{x+y}{z} + 8\prod\frac{z}{x+y} \geq 7; \ \ (1)$$

Denote: $\frac{x+y}{z} = a, \frac{y+z}{x} = b, \frac{z+x}{y} = c$

$$(1) \Leftrightarrow a+b+c+\frac{8}{abc} \geq 7; (2)$$

But $a+b+c \geq 3\sqrt[3]{abc}$ then is enough to prove

$$3\sqrt[3]{abc} + \frac{8}{abc} \geq 7$$

$$abc \geq 8 \ (Cesaro) \Rightarrow \sqrt[3]{abc} \geq 2$$

Let $\sqrt[3]{abc} = t \geq 2$, hence $3t + \frac{8}{t^3} \geq 7 \Leftrightarrow 3t^4 + 8 \geq 7t^3 \Leftrightarrow$

$$3t^4 - 6t^3 - t^3 + 2^3 \geq 0 \Leftrightarrow (t-2)(3t^3 - t^2 - 2t - 4) \geq 0$$

$$\Leftrightarrow (t-2)[(t-2)(3t^2+5t+8)+12] \geq 0, \forall t \geq 2$$

Solution 4 by Tran Hong-Dong Thap-Vietnam

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Let $t = \frac{x+y+z}{\sqrt[3]{xyz}} \overset{AM-GM}{\geq} 3$

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \overset{AM-GM}{\geq} (x+y+z)\frac{3}{\sqrt[3]{xyz}} = 3t; (1)$$

$$(x+y)(y+z)(z+x) \overset{AM-GM}{\leq} \frac{(2x+2y+2z)^3}{27} = \frac{8(x+y+z)^3}{27} \Rightarrow$$

$$\frac{8xyz}{(x+y)(y+z)(z+x)} \geq \frac{27xyz}{(x+y+z)^3} = \frac{27}{t^3}; (2)$$

From (1), (2) we need to prove:

$$3t + \frac{27}{t^3} \geq 10 \Leftrightarrow 3t^4 - 10t^3 + 27 \geq 0 \Leftrightarrow (t-3)(3t^3 - t^2 - 3t - 9) \geq 0$$

Which is true because:

$$t \geq 3 \Rightarrow t - 3 \geq 0; 3t^3 - t^2 - 3t - 9 = t^3 - t^2 + 2t^3 - 3t - 9 =$$

$$= t^2(t-1) + t(2t^2 - 3) - 9 > 0, \forall t \geq 3.$$

Solution 5 by Remus Florin Stanca-Romania

Let $x+y+z = s, xy+yz+zx = q, xyz = p$

$$(x+y+z\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = (x+y+z)\frac{xy+yz+zx}{xyz} = \frac{sq}{p}$$

$$(x+y+z)(xy+yz+zx) = x^2y + x^2z + xyz + xy^2 + xyz + y^2z + xyz + xz^2 + yz^2$$

$$= (x+y)(y+z)(z+x) + xyz \Rightarrow$$

$$sq = (x+y)(y+z)(z+x) + p \Rightarrow (x+y)(y+z)(z+x) = sq - p$$

The inequality can be written as:

$$\frac{sq}{p} = \frac{8p}{sq-p} \geq 10 \Rightarrow \frac{sq-p}{p} + \frac{8p}{sq-p} \geq 9$$

$$\text{Let } \frac{sq-p}{p} = a \Leftrightarrow a + \frac{8}{a} \geq 9$$

We prove that $(x+y+z)(xy+yz+zx) \geq 9xyz$:

$$x+y+z \geq 3\sqrt[3]{xyz}$$

$$xy+yz+zx \geq 3\sqrt[3]{x^2y^2z^2} \overset{(\cdot)}{\Rightarrow} (x+y+z)(xy+yz+zx) \geq 9xyz$$

$$sq \geq 9p \Leftrightarrow sq - p \geq 0 \Rightarrow a > 0 \Rightarrow a + \frac{8}{a} \geq 9 \Leftrightarrow a^2 - 9a + 8 \geq 0 \Rightarrow a_1 = 1, a_2 = 8$$

We prove that:

$$\frac{sq-p}{p}\geq 8\Leftrightarrow sq\geq 9p(true)\Rightarrow a^2-9a+8\geq 0$$

Therefore,

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 10$$

Solution 6 by Dang Le Gia Khanh-Vietnam

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\overset{AM-GM}{\geq}9;\ \ (1)$$

Suppose y lies between x and z $\Rightarrow (y-x)(y-z)\leq 0$

$$y^2+xz\leq xy+yz=y(x+z)$$

$$(y+x)(y+z)\leq 2y(x+z)\Leftrightarrow \frac{2y}{(x+y)(y+z)}\geq\frac{1}{x+z}$$

So,

$$\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{4xz}{(x+z)^2}$$

We will to prove:

$$\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{4xz}{(x+z)^2}\geq 2\Leftrightarrow$$

$$\left(y(x+z)-x^2-z^2\right)^2\geq 0\ (true)$$

Hence,

$$\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 2;\ \ (2)$$

From $(1),(2)$ we get:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 10$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.