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ROMANIAN MATHEMATICAL MAGAZINE

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If $x, y, z > 0$ then:

$$(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+y}\right)+\frac{4xyz}{(x+y)(y+z)(z+x)}\geq 5$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Daniel Sitaru-Romania, Solution 2 by Tran Hong-Dong Thap-Vietnam, Solution 3 by Hesenov Eldeniz-Georgia

Solution 1 by Daniel Sitaru-Romania

In ΔABC, we have:

$$s^2 \geq 16Rr - 5r^2 (GERRETSEN) \to s^2 + r^2 + 4r^2 \geq 16Rr \to$$

$$\frac{s^2+r^2}{4Rr}+\frac{r}{R}\geq 4 \to \frac{s^2+r^2}{4Rr}+1+\frac{r}{R}\geq 5 \to$$

$$\frac{s^2+r^2+4Rr}{4Rr}+\frac{r}{R}\geq 5 \to \frac{ab+bc+ca}{4Rr}+\frac{r}{R}\geq 5 \to$$

$$\frac{s(ab+bc+ca)}{4Rrs}+\frac{r}{R}\geq 5 \to \frac{s(ab+bc+ca)}{4RF}+\frac{r}{R}\geq 5 \to$$

$$s\cdot\frac{ab+bc+ca}{abc}+\frac{r}{R}\geq 5\to s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{r}{R}\geq 5\to$$

$$a=y+z,b=z+x,c=x+y,s=x+y+z,\frac{r}{R}=\frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+y}\right)+\frac{4xyz}{(x+y)(y+z)(z+x)}\geq 5$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)+\frac{4xyz}{(x+y)(y+z)(z+x)}\geq 5\Leftrightarrow$$

$$(x+y+z)\frac{(x+y)(y+z)+(y+z)(z+x)+(z+x)(x+y)}{(x+y)(y+z)(z+x)}+\frac{4xyz}{(x+y)(y+z)(z+x)}\geq 5$$

$$(x+y+x)[(x+y)(y+z)+(y+z)(z+x)+(z+x)(x+y)]+4xyz$$
$$\geq 5(x+y)(y+z)(z+x);\ \ (1)$$

Let $p=x+y+z, q=xy+yz+zx, r=xyz$

$$(1)\Leftrightarrow p(p^2+q)+4r\geq 5(pq-r)\Leftrightarrow p^3+9r\geq 4pq$$

Which is true by Schur's inequality:

$$a^3+b^3+c^3+3abc\geq ab(a+b)+bc(b+c)+ca(c+a)$$

$\Leftrightarrow p^3-3pq+3r+3r\geq pq-3r\Leftrightarrow p^3+9r\geq 4pq\Rightarrow(1)$ is true.

Solution 3 by Hesenov Eldeniz-Georgia

$$x=s-a, y=s-b, z=s-c\Rightarrow x+y+z=s$$

$$(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)+\frac{4xyz}{(x+y)(y+z)(z+x)}=$$

$$s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{4s(s-a)(s-b)(s-c)}{abc\cdot s}=$$

$$=s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{4s^2r^2}{4s^2rR}=s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{r}{R}\overset{(1)}{\geq}5$$

$$s\frac{s^2+r^2+4Rr}{abc}\overset{Gerretsen}{\geq}\frac{20Rr-4r^2}{4Rr}=5-\frac{r}{R}$$

$(1)\Leftrightarrow 5-\frac{r}{R}+\frac{r}{R}\geq 5$ true.

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.