R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

If $x_1, x_2, \dots, x_n > 0$ such that $x_1 + x_2 + \dots + x_n = n, n \ge 2, \lambda > 0$ then:

$$\frac{n\sqrt{x_1}+\lambda}{n\left(\sqrt{x_2}+\sqrt{x_3}+\dots+\sqrt{x_n}\right)}+\frac{n\sqrt{x_2}+\lambda}{n\left(\sqrt{x_1}+\sqrt{x_3}+\dots+\sqrt{x_n}\right)}+\dots$$
$$+\frac{n\sqrt{x_n}+\lambda}{n\left(\sqrt{x_1}+\sqrt{x_2}+\dots+\sqrt{x_{n-1}}\right)}\ge\frac{\sqrt{x_1}+\sqrt{x_2}+\dots+\sqrt{x_n}+\lambda}{n-1}$$

Proposed by Florică Anastase-Romania

Solution 1 by Michael Sterghiou-Greece, Solution 2 by Remus Florin Stanca-Romania, Solution 3 by proposer

Solution 1 by Michael Sterghiou-Greece

$$\sum_{i=1}^{n}\frac{n\sqrt{x_i}+\lambda}{n\sum_{j=1,j\neq i}^{n}\sqrt{x_j}}\ge\frac{\lambda+\sum_{i=1}^{n}\sqrt{x_i}}{n-1};\ (1)$$

$$\text{Let } \sqrt{x_i}=a_i\Rightarrow\sum_{i=1}^{n}a_i^2=n\Rightarrow n=\sum_{i=1}^{n}a_i^2\ge n\left(\frac{\sum_{i=1}^{n}a_i}{n}\right)^2\Rightarrow p=\sum_{i=1}^{n}a_i\le n.$$

Consider the function:

$$f(\lambda)=\sum_{i=1}^{n}\frac{na_i+\lambda}{n(p-a_i)}-\frac{\lambda+p}{n-1},$$

$$f'(\lambda)=\sum_{i=1}^{n}\frac{1}{n(p-a_i)}-\frac{1}{n-1}\overset{BCS}{\ge}\frac{n^2}{n\sum_{i=1}^{n}(p-a_i)}-\frac{1}{n-1}=\frac{n-p}{p(n-1)}\ge 0, p\le n$$

$$\Rightarrow f(\lambda)\nearrow, f(\lambda) = \lim_{\lambda\nearrow 0^+} f(\lambda) = \sum_{i=1}^{n}\frac{a_i}{p-a_i} - \frac{p}{n-1}$$

Which suffices to be positive. Consider the function $g(t) = \frac{t}{p-t}; g: (0,p) \to \mathbb{R}^+$

$$g''(t) = \frac{2p}{(p-t)^3} > 0 \Rightarrow g \text{ -convex.}$$

$$\sum_{i=1}^{n}\frac{a_i}{p-a_i} \ge n\frac{\frac{p}{n}}{p-\frac{p}{n}}$$

Hence, we need to prove:

$n\frac{\frac{p}{n}}{p-\frac{p}{n}} - \frac{p}{n-1} \ge 0$ which reduces to $\frac{n-p}{n-1} \ge 0$ and obviously holds as $n \ge p$.

Equality holds if and only if $x_1 = x_2 = \cdots = x_n, p = n$.

Solution 2 by Remus Florin Stanca-Romania

Let $s = \sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \Rightarrow$

$$\frac{\sqrt{x_1}}{\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n}} + \frac{\sqrt{x_2}}{\sqrt{x_1}+\sqrt{x_3}+\cdots+\sqrt{x_n}} + \cdots + \frac{\sqrt{x_n}}{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{n-1}}} =$$

$$= \frac{\sqrt{x_1}}{s-\sqrt{x_1}} + \frac{\sqrt{x_2}}{s-\sqrt{x_2}} + \cdots + \frac{\sqrt{x_n}}{s-\sqrt{x_n}} = s\left(\frac{1}{s-\sqrt{x_1}} + \frac{1}{s-\sqrt{x_2}} + \cdots + \frac{1}{s-\sqrt{x_n}}\right) - n; (1)$$

$$\left(\frac{1}{s-\sqrt{x_1}} + \frac{1}{s-\sqrt{x_2}} + \cdots + \frac{1}{s-\sqrt{x_n}}\right)(ns - s) \ge n^2 \Rightarrow$$

$$s\left(\frac{1}{s-\sqrt{x_1}} + \frac{1}{s-\sqrt{x_2}} + \cdots + \frac{1}{s-\sqrt{x_n}}\right) \ge \frac{n^2}{n-1} \overset{(1)}{\Rightarrow}$$

$$\frac{\sqrt{x_1}}{\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n}} + \frac{\sqrt{x_2}}{\sqrt{x_1}+\sqrt{x_3}+\cdots+\sqrt{x_n}} + \cdots + \frac{\sqrt{x_n}}{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{n-1}}}$$

$$\ge \frac{n^2}{n-1} - n = \frac{n}{n-1} \Rightarrow$$

$$\frac{n\sqrt{x_1}}{n(\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n})} + \frac{n\sqrt{x_2}}{n(\sqrt{x_1}+\sqrt{x_3}+\cdots+\sqrt{x_n})} + \cdots$$

$$+ \frac{n\sqrt{x_n}}{n(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{n-1}})} \ge \frac{n}{n-1}; (2)$$

$$\sum\frac{1}{\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n}}\ge\frac{n^2}{(n-1)s}\Rightarrow\frac{\lambda}{n}\sum\frac{\sqrt{x_1}}{\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n}}\ge\frac{\lambda n}{(n-1)s};(3)$$

We prove that:

$$\frac{\lambda n}{(n-1)s}\ge\frac{s+\lambda-n}{n-1}\Leftrightarrow\frac{\lambda n}{s}\ge s+\lambda-n\Rightarrow s^2+s(\lambda-n)-\lambda n\le 0$$

$$\Delta=(\lambda+n)^2\Rightarrow s_1=n,s_2=-\lambda$$

$$s=\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}\le\frac{x_1+x_2+\cdots+x_n+n}{2}=\frac{2n}{2}=n\Rightarrow$$

$$s\in[-\lambda,n]\Leftrightarrow s^2+s(\lambda-n)-\lambda n\le 0\overset{(3)}{\Rightarrow}$$

$$\frac{\lambda}{n(\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n})}+\frac{\lambda}{n(\sqrt{x_1}+\sqrt{x_3}+\cdots+\sqrt{x_n})}+\cdots+\frac{\lambda}{n(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{n-1}})}\ge\frac{s+\lambda-n}{n-1};(4)$$

From $(2),(4)$ we have:

$$\frac{n\sqrt{x_1}+\lambda}{n(\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n})}+\frac{n\sqrt{x_2}+\lambda}{n(\sqrt{x_1}+\sqrt{x_3}+\cdots+\sqrt{x_n})}+\cdots+\frac{n\sqrt{x_n}+\lambda}{n(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{n-1}})}\ge\frac{s+\lambda-n+n}{n-1}=\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+\lambda}{n-1}$$

Solution 3 by proposer

$$\frac{n\sqrt{x_1}+\lambda}{n^2(\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n})}+\frac{n\sqrt{x_2}+\lambda}{n^2(\sqrt{x_1}+\sqrt{x_3}+\cdots+\sqrt{x_n})}+\cdots+\frac{n\sqrt{x_n}+\lambda}{n^2(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{n-1}})}\ge\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+\lambda}{(n-1)(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n})}$$

Hence,

$$\frac{n\frac{\sqrt{x_1}}{\lambda}+1}{\frac{n^2}{\lambda}(\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n})}+\frac{n\frac{\sqrt{x_2}}{\lambda}+1}{\frac{n^2}{\lambda}(\sqrt{x_1}+\sqrt{x_3}+\cdots+\sqrt{x_n})}+\cdots+\frac{n\frac{\sqrt{x_n}}{\lambda}+1}{\frac{n^2}{\lambda}(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{n-1}})}\ge\frac{\frac{\sqrt{x_1}}{\lambda}+\frac{\sqrt{x_2}}{\lambda}+\cdots+\frac{\sqrt{x_n}}{\lambda}+1}{\frac{(n-1)}{\lambda}(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n})}$$

Let: $a_1 = \frac{\sqrt{x_1}}{\lambda}, a_2 = \frac{\sqrt{x_2}}{\lambda}, \dots, a_n = \frac{\sqrt{x_n}}{\lambda}, a_1, a_2, \dots, a_n > 0, n \ge 2$

We get:

$$\frac{na_1+\lambda}{n^2(a_2+a_3+\cdots+a_n)}+\frac{na_2+\lambda}{n^2(a_1+a_3+\cdots+a_n)}+\cdots+\frac{na_n+\lambda}{n^2(a_1+a_2+\cdots+a_{n-1})}\ge$$

$$\ge\frac{a_1+a_2+\cdots+a_n+\lambda}{(n-1)(a_1+a_2+\cdots+a_n)};\quad (*)$$

Denote $S = a_1+a_2+\cdots+a_n$**, we have:**

$$\frac{a_1}{S-a_1}+\frac{a_2}{S-a_2}+\cdots+\frac{a_n}{S-a_n}=\frac{a_1^2}{a_1(S-a_1)}+\frac{a_2^2}{a_2(S-a_2)}+\cdots+\frac{a_n^2}{a_n(S-a_n)}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(a_1+a_2+\cdots+a_n)}{S^2-(a_1^2+a_2^2+\cdots+a_n^2)}=\frac{S^2}{S^2-(a_1^2+a_2^2+\cdots+a_n^2)}$$

$$\frac{S^2}{S^2-(a_1^2+a_2^2+\cdots+a_n^2)}\ge\frac{n}{n-1}\Leftrightarrow(n-1)S^2\ge nS^2-n(a_1^2+a_2^2+\cdots+a_n^2)\Leftrightarrow$$

$$n(a_1^2+a_2^2+\cdots+a_n^2)\ge S^2 \text{ true form CBS.}$$

Now,

$$\frac{na_1}{n^2(a_2+a_3+\cdots+a_n)}+\frac{na_2}{n^2(a_1+a_3+\cdots+a_n)}+\cdots+\frac{na_n}{n^2(a_1+a_2+\cdots+a_{n-1})}$$

$$\ge\frac{1}{n-1};(1)$$

$$\frac{1}{S-a_1}+\frac{1}{S-a_2}+\cdots+\frac{1}{S-a_n}\overset{CBS}{\ge}\frac{n^2}{S-a_1+S-a_2+\cdots+S-a_n}=\frac{n^2}{(n-1)S}$$

$$\frac{\lambda}{n^2(a_2+a_3+\cdots+a_n)}+\frac{\lambda}{n^2(a_1+a_3+\cdots+a_n)}+\cdots+\frac{\lambda}{n^2(a_1+a_2+\cdots+a_{n-1})}\ge$$

$$\ge\frac{\lambda}{n^2}\cdot\frac{n^2}{(n-1)S}=\frac{\lambda}{(n-1)S};\quad(2)$$

From $(1)\&(2)$ **we get:**

$$\frac{na_1+\lambda}{n^2(a_2+a_3+\cdots+a_n)}+\frac{na_2+\lambda}{n^2(a_1+a_3+\cdots+a_n)}+\cdots+\frac{na_n+\lambda}{n^2(a_1+a_2+\cdots+a_{n-1})}\ge$$

$$\ge\frac{1}{n-1}+\frac{\lambda}{(n-1)S}=\frac{a_1+a_2+\cdots+a_n+\lambda}{(n-1)(a_1+a_2+\cdots+a_n)}$$

But

$$\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}}{n}\le\sqrt{\frac{x_1+x_2+\cdots+x_n}{n}}$$

Hence, we have:

$$\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+4\lambda}{(n-1)\left(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}\right)}\ge\frac{\sqrt{n}}{n-1}\cdot\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+4\lambda}{n\sqrt{x_1+x_2+\cdots+x_n}}\Leftrightarrow$$

$$\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+4\lambda}{(n-1)\left(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}\right)}\ge\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+4\subset}{(n-1)\sqrt{n(x_1+x_2+\cdots+x_n)}}\Leftrightarrow$$

$$\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+4\lambda}{(n-1)\left(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}\right)}\ge\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+4\lambda}{n(n-1)}$$

Therefore,

$$\frac{n\sqrt{x_1}+4\lambda}{n\left(\sqrt{x_2}+\sqrt{x_3}+\cdots+\sqrt{x_n}\right)}+\frac{n\sqrt{x_2}+4\lambda}{n\left(\sqrt{x_1}+\sqrt{x_3}+\cdots+\sqrt{x_n}\right)}+\cdots$$

$$+\frac{n\sqrt{x_n}+4\lambda}{n\left(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{n-1}}\right)}\ge\frac{\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}+4\lambda}{n-1}$$