R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\left(1+\frac{\sin^2 x}{\cos^2 y}\right)^{\cos^2 y}\cdot\left(1+\frac{\cos^2 x}{\sin^2 y}\right)^{\sin^2 y}\le 2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand, Solution 3 by Remus Florin Stanca-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\left(1+\frac{\sin^2 x}{\cos^2 y}\right)^{\cos^2 y}\cdot\left(1+\frac{\cos^2 x}{\sin^2 y}\right)^{\sin^2 y}=$$

$$=\left[\left(1+\frac{\sin^2 x}{\cos^2 y}\right)^{\cos^2 y}\cdot\left(1+\frac{\cos^2 x}{\sin^2 y}\right)^{\sin^2 y}\right]^{\frac{1}{\cos^2 y+\sin^2 y}}\overset{AM-GM}{\le}$$

$$\le\frac{1}{\sin^2 y+\cos^2 y}\left[\cos^2 y\left(1+\frac{\sin^2 x}{\cos^2 y}\right)+\sin^2 y\left(1+\frac{\cos^2 x}{\sin^2 y}\right)\right]=2$$

Therefore,

$$\left(1+\frac{\sin^2 x}{\cos^2 y}\right)^{\cos^2 y}\cdot\left(1+\frac{\cos^2 x}{\sin^2 y}\right)^{\sin^2 y}\le 2$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\left(1+\frac{sin^2x}{cos^2y}\right)^{cos^2y}\cdot\left(1+\frac{cos^2x}{sin^2y}\right)^{sin^2y}\le$$

$$\le\left(\frac{cos^2y\left(1+\frac{sin^2x}{cosy^2}\right)+sin^2y\left(1+\frac{cos^2x}{sin^2y}\right)}{cos^2y+sin^2y}\right)^{cos^2y+sin^2y}=$$

$$=\left(\frac{cos^2y+sin^2y+cos^2x+sin^2x}{cos^2y+sin^2y}\right)^{cos^2y+sin^2y}=\left(\frac{1+1}{1}\right)^1=2$$

Therefore,

$$\left(1+\frac{sin^2x}{cos^2y}\right)^{cos^2y}\cdot\left(1+\frac{cos^2x}{sin^2y}\right)^{sin^2y}\le 2$$

Solution 3 by Remus Florin Stanca-Romania

We apply Jensen inequality for a concave function on $I\subseteq\mathbb{R}, f:I\to\mathbb{R}$

If f is concave function on $I\Rightarrow\forall t_1,t_2\in(0,1)$ such that $t_1+t_2=1$ then

$$\forall x,y\in I\Rightarrow t_1f(x)+t_2f(y)\le f(t_1x+t_2y)$$

Let $f:(0,\infty)\to\mathbb{R}, f(x)=logx, \frac{\partial^2 f}{\partial x^2}=-\frac{1}{x^2}\le 0\Rightarrow f$ –concave, hence

$$log\left[\left(1+\frac{sin^2x}{cos^2y}\right)^{cos^2y}\cdot\left(1+\frac{cos^2x}{sin^2y}\right)^{sin^2y}\right]=$$

$$=cos^2y\cdot log\left(1+\frac{sin^2x}{cos^2y}\right)+sin^2y\cdot log\left(1+\frac{cos^2x}{sin^2y}\right)=$$

$$=cos^2y\cdot log\left(1+\frac{sin^2x}{cos^2y}\right)+sin^2y\cdot log\left(1+\frac{cos^2x}{sin^2y}\right)\le$$

$$\overset{Jensen}{\le} log\left[cos^2y\left(1+\frac{sin^2x}{cos^2y}\right)+sin^2y\left(1+\frac{cos^2x}{sin^2y}\right)\right]=$$

$$=log(sin^2x+cos^2x+sin^2y+cos^2y)=log2$$

Therefore,

$$\left(1+\frac{sin^2x}{cos^2y}\right)^{cos^2y}\cdot\left(1+\frac{cos^2x}{sin^2y}\right)^{sin^2y}\le 2$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.