R M M

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If $x, y, z > 0, x^6 + y^6 + z^6 = x^4 + y^4 + z^4$ then:

$$\frac{x^4}{y^2 + yz + z^2} + \frac{y^4}{z^2 + zx + x^2} + \frac{z^4}{x^2 + xy + y^2} \geq 1$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Abdul Hannan-Tezpur-India

Recall: $$\sum_{cyc} \frac{a}{b+c} - \frac{3}{2} = \frac{1}{2}\sum_{cyc} \frac{(a-b)^2}{(b+c)(c+a)} = \frac{\sum_{cyc}(a+b)(a-b)^2}{2(a+b)(b+c)(c+a)}$$

Note that: $$\sum_{cyc} \frac{x^4}{y^2 + yz + z^2} \overset{AGM}{\geq} \sum_{cyc} \frac{x^4}{y^2 + \frac{y^2+z^2}{2} + z^2} = \frac{2}{3}\sum_{cyc} \frac{x^4}{y^2 + z^2}$$

We will show that:

$$\frac{2}{3}\sum_{cyc} \frac{x^4}{y^2 + z^2} \geq \frac{x^6 + y^6 + z^6}{x^4 + y^4 + z^4} = 1$$

Let $a = x^2, b = y^2, c = z^2$. Then for $a, b, c > 0$ we will prove that:

$$\frac{2}{3}\sum_{cyc} \frac{a^2}{b+c} \geq \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2} \Leftrightarrow$$

$$\frac{2}{3}\left[\sum_{cyc}\frac{a^2}{b+c}-\frac{a+b+c}{2}\right]\ge\frac{a^3+b^3+c^3}{a^2+b^2+c^2}-\frac{a+b+c}{3}$$

$$\Leftrightarrow\frac{2}{3}\left[\sum_{cyc}\left(\frac{a^2}{b+c}-a\right)-\frac{a+b+c}{2}\right]\ge\frac{3(a^3+b^3+c^3)-(a+b+c)(a^2+b^2+c^2)}{3(a^2+b^2+c^2)}$$

$$\Leftrightarrow\frac{2}{3}\left[(a+b+c)\sum_{cyc}\frac{a}{b+c}-\frac{3(a+b+c)}{2}\right]$$

$$\ge\frac{2(a^3+b^3+c^3)-ab(a+b)-bc(b+c)0ca(c+a)}{3(a^2+b^2+c^2)}$$

$$\Leftrightarrow\frac{2(a+b+c)}{3}\left(\frac{\sum_{cyc}(a+b)(a-b)^2}{2(a+b)(b+c)(c+a)}\right)\ge\frac{\sum_{cyc}(a+b)(a-b)^2}{3(a^2+b^2+c^2)}$$

using the remark above.

$$\Leftrightarrow\left[\frac{\sum_{cyc}(a+b)(a-b)^2}{3}\right]\left[\frac{a+b+c}{(a+b)(b+c)(c+a)}-\frac{1}{a^2+b^2+c^2}\right]\ge 0$$

$$\Leftrightarrow\left[\frac{\sum_{cyc}(a+b)(a-b)^2}{3}\right]\cdot\frac{(a+b+c)(a^2+b^2+c^2)-(a+b)(b+c)(c+a)}{(a+b)(b+c)(c+a)(a^2+b^2+c^2)}\ge 0$$

So, it is enough to prove that

$$(a+b+c)(a^2+b^2+c^2)-(a+b)(b+c)(c+a)\ge 0$$

True, because

$$(a+b+c)(a^2+b^2+c^2)-(a+b)(b+c)(c+a)$$

$$\ge(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a)=abc>0$$

Note by editor:

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