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ABOUT AN INEQUALITY BY NGUYEN VAN NHO-I

By Marin Chirciu – Romania

1) Let $a,b,c>0$. Prove that:

$$\sum\frac{b+c}{a+\sqrt{(a+2b)(a+2c)}}\ge\frac{3}{2}$$

Proposed by Nguyen Van Nho - Vietnam

Solution

Using AM-GM and Bergström we obtain:

$$\sum\frac{b+c}{a+\sqrt{(a+2b)(a+2c)}}\ge\sum\frac{b+c}{a+\frac{(a+2b)+(a+2b)}{2}}=\sum\frac{b+c}{2a+b+c}=$$

$$=\sum\frac{(b+c)^2}{(b+c)(2a+b+c)}\ge\frac{(\sum(b+c))^2}{\sum(b+c)(2a+b+c)}=\frac{4(\sum a^2+2\sum bc)}{2\sum a^2+6\sum bc}=\frac{2(\sum a^2+2\sum bc)}{\sum a^2+3\sum bc}\ge\frac{3}{2},$$ *where the last inequality is equivalent with* $\sum a^2\ge\sum bc\Leftrightarrow\sum(a-b)^2\ge 0$, *obviously with equality for* $a=b=c$.

Remark. *The inequality can be developed.*

2) Let $a,b,c>0$ and $0\le n\le 4$. Prove that:

$$\sum\frac{b+c}{a+\sqrt{(a+nb)(a+nc)}}\ge\frac{6}{n+2}$$

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Solution *Using AM-GM and Bergström's inequality we obtain:*

$$\sum\frac{b+c}{a+\sqrt{(n+ab)(a+nc)}}\ge\sum\frac{b+c}{a+\frac{(a+nb)+(a+nb)}{2}}=2\sum\frac{b+c}{4a+n(b+c)}=$$

$$=2\sum\frac{(b+c)^2}{(b+c)\big(4a+n(b+c)\big)}\ge\frac{2(\sum(b+c))^2}{\sum(b+c)\big(4a+n(b+c)\big)}=\frac{8(\sum a^2+2\sum bc)}{2n\sum a^2+(2n+8)\sum bc}$$

$$=\frac{4(\sum a^2+2\sum bc)}{n\sum a^2+(n+4)\sum bc}\ge\frac{6}{n+2},$$ *where the last inequality is equivalent with*

$(4-n)\sum a^2\ge(4-n)\sum bc\Leftrightarrow(4-a)\sum(a-b)^2\ge 0$, *obviously, because* $0\le n\le 4$ *and*

$\sum(a-b)^2 \geq 0$. *We deduce that the proposed inequality holds, with equality for* $a=b=c$ *or* $n=0$.

Note.*For* $n=2$ *we obtain problem VIII.4 from RMM-24, Spring 2020, proposed by Nguyen Van Nho, Vietnam. In the same way, we propose:*

3) Let be $a,b,c>0$ **and** $k\geq 0$**. Prove that:**

$$\sum \frac{b+c}{ka+\sqrt{(a+2b)(a+2c)}} \geq \frac{6}{k+3}$$

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Solution *Using AM-GM and Bergström's inequality, we obtain:*

$$\sum \frac{b+c}{ka+\sqrt{(a+2b)(a+2c)}} \geq \sum \frac{b+c}{ka+\frac{(a+2b)+(a+2b)}{2}} = \sum \frac{b+c}{ka+a+b+c} =$$

$$= \sum \frac{(b+c)^2}{(b+c)(ka+a+b+c)} \geq \frac{(\sum(b+c))^2}{\sum(b+c)(ka+a+b+c)} = \frac{4(\sum a^2+2\sum bc)}{2k\sum a^2+(2k+4)\sum bc} =$$

$$= \frac{2(\sum a^2+2\sum bc)}{k\sum a^2+(k+2)\sum bc} \geq \frac{6}{k+3},$$ *where the last inequality is equivalent with*

$k\sum a^2 \geq k\sum bc \Leftrightarrow k\sum(a-b)^2 \geq 0$, *obviously with equality for* $a=b=c$ *or* $k=0$.

Note. *For* $k=0$ *we obtain problem VIII.4 from RMM-24, Spring 2020, proposed by Nguyen Van Nho, Vietnam.*

4) Let be $a,b,c>0, k\geq 0$ **and** $0\leq n\leq 2k+2$**. Prove that:**

$$\sum \frac{b+c}{ka+\sqrt{(a+nb)(a+nc)}} \geq \frac{6}{k+n+1}$$

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Solution *Using AM-GM and Bergström's inequality we obtain:*

$$\sum \frac{b+c}{ka+\sqrt{(a+nb)(a+nc)}} \geq \sum \frac{b+c}{ka+\frac{(a+nb)+(a+nb)}{2}} =$$

$$= 2\sum \frac{b+c}{2ka+2a+n(b+c)} = 2\sum \frac{(b+c)^2}{(b+c)\big(2ka+2a+n(b+c)\big)} \geq$$

$$\ge \frac{2\left(\sum(b+c)\right)^2}{\sum(b+c)\big(2ka+2a+n(b+c)\big)} = \frac{8\left(\sum a^2+2\sum bc\right)}{2n\sum a^2+(4k+2n+4)\sum bc} =$$

$= \frac{4\left(\sum a^2+2\sum bc\right)}{n\sum a^2+(2k+n+2)\sum bc} \ge \frac{6}{k+n+1}$, *where the last inequality is equivalent with*

$(2k+2-n)\sum a^2 \ge (2k+2-n)\sum bc \Leftrightarrow (2k+2-n)\sum(a-b)^2 \ge 0$, *obviously because*

$0 \le n \le 2k+2$ *and* $\sum(a-b)^2 \ge 0$.

We deduce that the proposed inequality holds, with equality for $a=b=c$ *or* $k=n=0$.

Note. *For* $k=1, n=2$ *we obtain problem VIII.4 from RMM – 24, Spring 2020, proposed by Nguyen Van Nho, Vietnam.*

Reference:

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