R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

# ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-VIII

***By Marin Chirciu – Romania***

**1) If $I$ – incenter $\Delta ABC$, then:**

$$[AIB][AIC]+[BIC]+[CIA][CIB]\le r^2(R+r)^2$$

***Proposed by Marian Ursărescu – Romania***

***Solution*** *We prove the following Lemma:*

***Lemma.***

***2) If I – incenter, ΔABC, then:***

$$[AIB][AIC]+[BIC][BIA]+[CIA][CIB]=\frac{r^2(s^2+r^2+4Rr)}{4}$$

***Proof.***

*We have* $[BIC]=\frac{ar}{2}$ *and the analogs. It follows*

$[AIB][AIC]+[BIC][BIA]+[CIA][CIB]=\frac{r^2}{4}(bc+ca+ab)=\frac{r^2(s^2+r^2+4Rr)}{4}$, *which follows from the known identity in triangle* $\sum bc=s^2+r^2+4Rr$. *Let's get back to the main problem. Using lemma the inequality can be written:*

$$\frac{r^2(s^2+r^2+4Rr)}{4}\le r^2(R+r)^2\Leftrightarrow s^2\le 4R^2+4Rr+3r^2 \text{ (Gerretsen's inequality)}$$

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*Let's get back to the main problem.*

***3) If I – incenter ΔABC, then:***

$$[AIB][AIC]+[BIC][BIA]+[CIA][CIB]\ge 9r^4$$

***Proposed by Marin Chirciu – Romania***

***Solution***

*Using lemma the inequality can be written:*

$\frac{r^2(s^2+r^2+4Rr)}{4}\ge 9r^4\Leftrightarrow s^2\ge 35r^2-4Rr$, *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$$16Rr - 5r^2 \geq 35r^2 - 4Rr \Leftrightarrow R \geq 2r \text{ (Euler's inequality).}$$

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*We can write the double inequality:*

**4) If $I$ – incenter $\Delta ABC$, then:**

$$9r^4 \leq [AIB][AIC] + [BIC][BIA] + [CIA][CIB] \leq r^2(R + r)^2$$

***Solution***

*See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.*

***Reference:***

***Romanian Mathematical Magazine-www.ssmrmh.ro***