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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-VI

By Marin Chirciu – Romania

1) Prove that in any acute-angled triangle the following inequality holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{12r^2}{R^2}$$

Proposed by Marian Ursărescu – Romania

Solution *We prove the following lemma:*

2) In ΔABC the following relationship holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{(\sum bc)^2}{4R^2s^2}$$

Proof.*Using the following formulas:* $h_a=\frac{2S}{a}$ *and* $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$*, we obtain:*

$$\frac{h_a^2}{w_a^2}=\frac{\left(\frac{2S}{a}\right)^2}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}=\frac{S^2}{a^2b^2c^2}\cdot\frac{bc(b+c)^2}{s(s-a)}\ge\frac{S^2}{16R^2S^2}\cdot\frac{bc\cdot 4bc}{s(s-a)}=\frac{b^2c^2}{4R^2s(s-a)}$$

It follows $\sum\frac{h_a^2}{w_a^2}\ge\frac{1}{4R^2}\sum\frac{b^2c^2}{s(s-a)}\overset{Bergstrom}{\ge}\frac{1}{4R^2}\cdot\frac{(\sum bc)^2}{\sum s(s-a)}=\frac{(\sum bc)^2}{4R^2s^2}$

Let's get back to the main problem:

Using the Lemma it suffices to prove that:

$$\frac{(\sum bc)^2}{4R^2s^2}\ge\frac{12r^2}{R^2}\Leftrightarrow\frac{(s^2+r^2+4Rr)^2}{4R^2s^2}\ge\frac{12r^2}{R^2}\Leftrightarrow s^2(s^2+8Rr-46r^2)+r^2(4R+r)^2\ge 0$$

We distinguish the following cases:

Case 1). If $(s^2+8Rr-46r^2)\ge 0$*, the inequality is obvious.*

Case 2). If $(s^2+8Rr-46r^2)<0$*, the inequality can be rewritten:*

$r^2(4R+r)^2\ge s^2(46r^2-8Rr-s^2)$*, which follows from Blundon-Gerretsen's inequality:*

$16Rr-5r^2\le s^2\le\frac{R(4R+R)^2}{2(2R-r)}$*. It remains to prove that:*

$$r^2(4R+r)^2\ge\frac{R(4R+r)^2}{2(2R-r)}(46r^2-8Rr-16Rr+5r^2)\Leftrightarrow 24R^2-47Rr-2r^2\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(4R+r)\ge 0$*, obviously from Euler's inequality* $R\ge 2r$*.*

Equality holds if and only if the triangle is equilateral.

Remark. *From the above proof, the condition of acute-angled triangle it is not necessary.*

Remark. *Inequality can be strengthened:*

3) In ΔABC the following inequality holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{6r}{R}$$

Proposed by Marin Chirciu – Romania

Solution *Using Lemma, it suffices to prove that:*

$$\frac{(\sum bc)^2}{4R^2s^2}\ge\frac{6r}{R}\Leftrightarrow\frac{(s^2+r^2+4Rr)^2}{4R^2s^2}\ge\frac{6r}{R}\Leftrightarrow s^2(s^2+2r^2-16Rr)+r^2(4R+r)^2\ge 0$$

We distinguish the following cases:

Case 1). If $(p^2+2r^2-16Rr)\ge 0$, the inequality is obvious.

Case 2). If $(s^2+2r^2-16Rr)<0$, the inequality can be rewritten:

$r^2(4R+r)^2\ge s^2(16Rr-2r^2-s^2)$, *which follows from Blundon-Gerretsen's inequality:*

$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}$. *It remains to prove that:*

$$r^2(4R+r)^2\ge\frac{R(4R+R)^2}{2(2R-r)}(16Rr-2r^2-16Rr+5r^2)\Leftrightarrow R\ge 2r\ (Euler)$$

Equality holds if and only if the triangle is equilateral.

Remark. *Inequality 3) is stronger than inequality 1)*

4) In ΔABC the following relationship holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\ge\frac{6r}{R}\ge\frac{12r^2}{R^2}$$

Solution

See inequality 3) and $\frac{6r}{R}\ge\frac{12r^2}{R^2}\Leftrightarrow R\ge 2r$ *(Euler)*

Equality holds if and only if the triangle is equilateral.

Remark.

If we replace h_a with r_a we propose:

5) In ΔABC the following relationship holds:

$$\frac{r_a^2}{w_a^2}+\frac{r_b^2}{w_b^2}+\frac{r_c^2}{w_c^2}\geq\frac{3R}{2r}$$

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Solution *We prove the following lemma:*

Lemma.

6) In ΔABC the following relationship holds:

$$\frac{r_a^2}{w_a^2}+\frac{r_b^2}{w_b^2}+\frac{r_c^2}{w_c^2}\geq\sum\frac{r^2s}{(s-a)^3}$$

Proof. *Using the following formulas:* $r_a=\frac{S}{s-a}$ *and* $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, *we obtain:*

$$\frac{r_a^2}{w_a^2}=\frac{\left(\frac{S}{s-a}\right)^2}{\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)^2}=\frac{S^2}{4s}\cdot\frac{(b+c)^2}{bc(s-a)^3}\geq\frac{r^2s^2}{4s}\cdot\frac{4bc}{bc(s-a)^3}=\frac{r^2s}{(s-a)^3}$$

It follows $\sum\frac{r_a^2}{w_a^2}\geq\sum\frac{r^2s}{(s-a)^3}$.

Let's get back to the main problem.

Using Lemma it suffices to prove that: $\sum\frac{r^2s}{(s-a)^3}\geq\frac{3R}{2r}$

Using the identity in triangle: $\sum\frac{1}{(s-a)^3}=\frac{(4R+r)^3-12s^2R}{r^3s^3}$ *the inequality holds:*

$\sum\frac{r^2s}{(s-a)^3}\geq\frac{3R}{2r}$ *we write* $r^2s\cdot\frac{(4R+r)^3-12s^2R}{r^3s^3}\geq\frac{3R}{2r}\Leftrightarrow 2(4R+r)^3-24s^2R\geq 3s^2R\Leftrightarrow$

$\Leftrightarrow 2(4R+r)^3\geq 27s^2R$, *it follows from Blundon-Gerretsen's inequality.*

$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$. *It remains to prove that:*

$$2(4R+r)^3\geq 27R\cdot\frac{R(4R+r)^2}{2(2R-r)}\Leftrightarrow 5R^2-8Rr-4r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(5R+2r)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Reference:

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