R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

ABOUT AN INEQUALITY BY JOSE LUIS DIAZ BARRERO-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\ge\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}}$$

Proposed by Jose Luis Diaz-Barrero - Spain

Proof. *We have*

$$2)\sum\frac{a^2+bc}{b+c}=\frac{\sum(a^2+bc)(a+b)(a+b)}{\prod(b+c)}=\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}$$

which follows from

$$\sum(a^2+bc)\,(a+b)(a+b)=5s^4-10s^2r^2+r^2(4R+r)^2 \text{ and}$$

$$\prod(b+c)=2s(s^2+r^2+2Rr)$$

Let's get back to the main problem.

Using the Lemma and the known inequalities in triangle $abc=4Rrs$ *and* $\prod h_a=\frac{2s^2r^2}{R}$

We prove the stronger inequality

3) In ΔABC the following inequality holds:

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\ge a+b+c$$

Solution $\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}\ge 2s\Leftrightarrow s^4-s^2(8Rr+14r^2)+r^2(4R+r)^2\ge 0\Leftrightarrow$

$$\Leftrightarrow s^2(s^2-8Rr-14r^2)+r^2(4R+r)^2\ge 0$$

We distinguish the following cases:

Case 1). If $(s^2-8Rr-14r^2)\ge 0$, *the inequality is obvious.*

Case 2). If $(s^2-8Rr-14r^2)<0$, *we rewrite the inequality:*

$r^2(4R+r)^2\ge s^2(8Rr+14r^2-s^2)$, *which follows from Gerretsen's inequality*

$$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2.$$

It remains to prove that:

$$r^2(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}(8Rr+14r^2-16Rr+5r^2) \Leftrightarrow$$

$$\Leftrightarrow 2r^2(2R-r) \leq R(-8Rr+19r^2) \Leftrightarrow 8R^2-15Rr-2r^2 \geq 0 \Leftrightarrow (R-2r)(8R+r) \geq 0,$$

obviously from Euler's inequality $R \geq 2r$.

4) In ΔABC the following inequality holds:

$$\boldsymbol{a+b+c \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}}}$$

Solution *We prove that:*

$$a+b+c \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}} \Leftrightarrow 2s \geq \frac{3 \cdot 4Rrs}{2R}\sqrt[3]{\frac{R}{2s^2r^2}} \Leftrightarrow 1 \geq 3r \cdot \sqrt[3]{\frac{R}{2s^2r^2}} \Leftrightarrow$$

$\Leftrightarrow 2s^2 \geq 27Rr$*, which follows from Gerretsen's inequality* $s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$$2(16Rr-5r^2) \geq 27Rr \Leftrightarrow R \geq 2r. \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

Remark. *We can write:*

5) In ΔABC the following relationship holds:

$$\boldsymbol{\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq a+b+c \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}}}$$

Solution *See 3) and 4).*

Equality holds if and only if the triangle is equilateral.

Remark. *Let's find an inequality having an opposite sense:*

6) In ΔABC the following relationship holds:

$$\boldsymbol{\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \leq (a+b+c)\frac{R}{2r}}$$

Proposed by Marin Chirciu – Romania

Solution *Using the Lemma, the inequality can be written:*

$$\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)} \le s\cdot\frac{R}{r} \Leftrightarrow s^2[s^2(2R-5r)+2r(2R^2+Rr+5r^2)] \ge r^3(4R+r)^2.$$

We distinguish the following cases:

Case 1). If $(2R-5r)\ge 0$, *we use Gerretsen's inequality* $s^2 \ge 16Rr-5r^2 \ge \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(2R-5r)+2r(2R^2+Rr+5r^2)] \ge r^3(4R+r)^2 \Leftrightarrow$$

$\Leftrightarrow 36R^2-89Rr+34r^2\ge 0$, *obviously, because in this case* $2R\ge 5r$.

Case 2). If $(2R-5r)<0$, *we rewrite the inequality:*

$s^2[2r(2R^2+Rr+5r^2)-s^2(5r-2R)] \ge r^3(4R+r)^2$, *and we use Gerretsen's inequality*

$$\frac{r(4R+r)^2}{R+r} \le 16Rr-5r^2 \le s^2 \le \frac{R(4R+r)^2}{2(2R-r)} \le 4R^2+4Rr+3r^2.$$

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[2r(2R^2+Rr+5r^2)-(4R^2+4Rr+3r^2)(5r-2R)] \ge r^3(4R+r)^2 \Leftrightarrow$$

$\Leftrightarrow 8R^3-8R^2r-13Rr^2-6r^3 \ge 0 \Leftrightarrow (R-2r)(8R^2+8Rr+3r^2)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

Remark. *We can write:*

7) In ΔABC the following relationship holds:

$$a+b+c \le \frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \le (a+b+c)\frac{R}{2r}$$

Proposed by Marin Chirciu – Romania

Solution *See 3) and 6). Equality holds if and only if the triangle is equilateral.*

8) In ΔABC the following relationship holds:

$$a+b+c \le \frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \le (a+b+c)\frac{R}{2r}.$$

Proposed by Marin Chirciu – Romania

Solution *We prove the following lemma:*

Lemma.

In ΔABC the following relationship holds:

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}=\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}$$

Proof.

We have $\sum\frac{a^2+bc}{b+c}=\frac{\sum(a^2+bc)(a+b)(a+b)}{\prod(b+c)}=\frac{5s^2-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}$, *which follows from*

$\sum(a^2+bc)\,(a+b)(a+b)=5s^4-10s^2r^2+r^2(4R+r)^2$ *and*

$\prod(b+c)=2s(s^2+r^2+2Rr)$.

Let's get to the main problem.

LHS inequality. Using the Lemma the inequality can be written:

$$\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}\geq 2s\Leftrightarrow s^4-s^2(8Rr+14r^2)+r^2(4R+r)^2\geq 0\Leftrightarrow$$

$$\Leftrightarrow s^2(s^2-8Rr-14r^2)+r^2(4R+r)^2\geq 0$$

We distinguish the following cases:

Case 1). If $(s^2-8Rr-14r^2)\geq 0$, *the inequality is obvious.*

Case 2). If $(s^2-8Rr-14r^2)<0$, *the inequality rewrites itself:*

$r^2(4R+R)^2\geq s^2(8R+14r^2-s^2)$, *which follows from Gerretsen's inequality:*

$$16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

It remains to prove that:

$$r^2(4R+r)^2\geq\frac{R(4R+r)^2}{2(2R-r)}(8Rr+14r^2-16Rr+5r^2)\Leftrightarrow$$

$$\Leftrightarrow 2r^2(2R-r)\leq R(-8Rr+19r^2)\Leftrightarrow$$

$\Leftrightarrow 8R^2-15Rr-2r^2\geq 0\Leftrightarrow(R-2r)(8R+r)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$. *Equality holds if and only if the triangle is equilateral.*

RHS inequality. Using the Lemma, the inequality rewrites itself:

$\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}\leq s\cdot\frac{R}{r}\Leftrightarrow s^2[s^2(2R-5r)+2r(2R^2+Rr+5r^2)]\geq r^3(4R+r)^2.$

We distinguish the following cases:

Case 1). If $(2R-5r)\geq 0$, *we use Gerretsen's inequality* $s^2\geq 16Rr-5r^2\geq \frac{r(4R+r)^2}{R+r}$

It remain to prove that:

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(2R-5r)+2r(2R^2+Rr+5r^2)]\geq r^3(4R+r)^2 \Leftrightarrow$$

$\Leftrightarrow 36R^2-89Rr+34r^2\geq 0$, *obviously, because in this case* $2R\geq 5r$.

Case 2). If $(2R-5r)<0$, *we rewrite the inequality*

$s^2[2r(2R^2+Rr+5r^2)-s^2(5r-2R)]\geq r^3(4R+r)^2$ *and we use Gerretsen's inequality*

$$\frac{r(4R+r)^2}{R+r}\leq 16Rr-5r^2\leq s^2\leq \frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[2r(2R^2+Rr+5r^2)-(4R^2+4Rr+3r^2)(5r-2R)]\geq r^3(4R+r)^2 \Leftrightarrow$$

$\Leftrightarrow 8R^3-8R^2r-13Rr^2-6r^3\geq 0 \Leftrightarrow (R-2r)(8R^2+8Rr+3r^2)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$. *Equality holds if and only if the triangle is equilateral.*

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro