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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-VI

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1.In ΔABC the following relationship holds:

$$\sum_{cyc} sin^2 A cos\frac{B}{2} \le 3\sqrt{3}\left(\frac{1}{2}-\frac{r^3}{R^3}\right)$$

Proposed by George Apostolopoulos-Greece

Solution by Marin Chirciu-Romania

$$\sum_{cyc} sin^2 A cos\frac{B}{2} \le 3\sqrt{3}\left(\frac{1}{2}-\frac{r^3}{R^3}\right) \Leftrightarrow \sum_{cyc}(1-cos^2 A)cos\frac{B}{2} \le 3\sqrt{3}\left(\frac{1}{2}-\frac{r^3}{R^3}\right) \Leftrightarrow$$

$\sum_{cyc} cos\frac{A}{2} - \sum_{cyc} cos^2 A cos\frac{B}{2} \le 3\sqrt{3}\left(\frac{1}{2}-\frac{r^3}{R^3}\right)$ which follows from inequalities:

$\sum_{cyc} cos\frac{A}{2} \le \frac{3\sqrt{3}}{2}$ and $\sum_{cyc} cos^2 A cos\frac{B}{2} \ge 3\sqrt{3}\frac{r^3}{R^3}$

Lemma 1. **2. In ΔABC the following relationship holds:**

$$cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2} \le \frac{3\sqrt{3}}{2}$$

R. Kooistra, Item 2.27, Geometric Inequality, O.Bottema,1969

Proof. Applying Jensen Inequality for concave function $x \to cos\, x, x \in (0,\pi)$, we get:

$$cos\frac{A}{2}+cos\frac{B}{2}+cos\frac{C}{2} \le 3cos\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3} = \frac{3\sqrt{3}}{2}$$

Equality holds if and only if triangle is equilateral.

Lemma 2. **3. In ΔABC the following relationship holds:**

$$\frac{1}{cos\frac{A}{2}}+\frac{1}{cos\frac{B}{2}}+\frac{1}{cos\frac{C}{2}} \le \frac{a^2+b^2+c^2}{2S}$$

Proof. We have:

$$2S\sum_{cyc}\frac{1}{cos\frac{A}{2}}=\sum_{cyc}\frac{bc\cdot sinA}{cos\frac{A}{2}}=\sum_{cyc}\frac{bc\cdot 2sin\frac{A}{2}cos\frac{A}{2}}{cos\frac{A}{2}}=2\sum_{cyc}bc\cdot sin\frac{A}{2}\overset{(1)}{\leq}a^2+b^2+c^2$$

Where, $(1)\Leftrightarrow\ 2\sum_{cyc}bc\cdot sin\frac{A}{2}\leq a^2+b^2+c^2\Leftrightarrow a^2-\ 2\sum_{cyc}bc\cdot sin\frac{A}{2}+b^2+c^2\geq 0$

$a^2-2a\left(bsin\frac{C}{2}+csin\frac{B}{2}\right)+b^2+c^2-2bc\cdot sin\frac{A}{2}\geq 0$ which follows from

$$\Delta=4\left(bsin\frac{C}{2}+csin\frac{B}{2}\right)^2-4\left(b^2+c^2-2bc\cdot sin\frac{A}{2}\right)=$$

$\overset{(2)}{=}-4\left(bcos\frac{C}{2}-ccos\frac{B}{2}\right)^2\leq 0$ where (2) $\Leftrightarrow cos\frac{B}{2}cos\frac{C}{2}=sin\frac{A}{2}+sin\frac{B}{2}sin\frac{C}{2}$.

Equality holds if and only if triangle is equilateral.Let's proof that inequality:

$\sum_{cyc}cos^2Acos\frac{B}{2}\geq 3\sqrt{3}\frac{r^3}{R^3}$. Using Bergstrom Inequality, we have:

$$\sum_{cyc}cos^2Acos\frac{B}{2}=\sum_{cyc}\frac{cos^2A}{\frac{1}{cos\frac{B}{2}}}\geq\frac{\left(\sum_{cyc}cosA\right)^2}{\sum_{cyc}\frac{1}{cos\frac{B}{2}}}=\frac{\left(1+\frac{r}{R}\right)^2}{\sum_{cyc}\frac{1}{cos\frac{A}{2}}}\overset{Lemma\ 2}{\geq}\frac{\left(1+\frac{r}{R}\right)^2}{\frac{a^2+b^2+c^2}{2S}}\overset{(2)}{\geq}$$

$\overset{(2)}{\geq}\ 3\sqrt{3}\frac{r^3}{R^3}$, where $(2)\Leftrightarrow\frac{\left(1+\frac{r}{R}\right)^2}{\frac{a^2+b^2+c^2}{2S}}\geq 3\sqrt{3}\frac{r^3}{R^3}\Leftrightarrow\frac{\frac{(R+r)^2}{R^2}}{\frac{2(s^2-r^2-4Rr)}{2sr}}\geq 3\sqrt{3}\frac{r^3}{R^3}\Leftrightarrow$

$$\frac{sr(R+r)^2}{R^2(s^2-r^2-4Rr)}\geq 3\sqrt{3}\frac{r^3}{R^3}\Leftrightarrow sR(R+r)^2\geq 3\sqrt{3}r^2(s^2-r^2-4Rr)$$

which result from $s\geq 3r\sqrt{3}$ (Mitrinovic) and $s^2\leq 4R^2+4Rr+3r^2$ (Gerretsen).

It remains to prove that:

$$3\sqrt{3}R(R+r)^2\geq 3\sqrt{3}r^2(4R^2+4Rr+3r^2-r^2-4Rr)\Leftrightarrow$$
$$R(R+r)^2\geq r(4R^2+2r^2)\Leftrightarrow R^3-2R^2r+Rr^2-2r^3\geq 0\Leftrightarrow$$

$(R-2r)(R^2+r^2)\geq 0$ which is true from $R\geq 2r$ (Euler).

Equality holds if and only if triangle is equilateral.

From $\sum_{cyc}cos\frac{A}{2}\leq\frac{3\sqrt{3}}{2}$ and $\sum_{cyc}cos^2Acos\frac{B}{2}\geq 3\sqrt{3}\frac{r^3}{R^3}$ we has conclusion.

Equality holds if and only if triangle is equilateral.

4. In ΔABC the following relationship holds:

$$\sum_{cyc}cos^2Asin\frac{B}{2}\leq 3\left(\frac{1}{2}-\frac{3r^3}{R^3}\right)$$

Proposed by Marin Chirciu-Romania

Solution by proposer Inequality it can be rewritten as:

$$\sum_{cyc} \cos^2 A \sin\frac{B}{2} \le 3\left(\frac{1}{2} - \frac{3r^3}{R^3}\right) \Leftrightarrow \sum_{cyc} (1 - \sin^2 A)\sin\frac{B}{2} \le 3\left(\frac{1}{2} - \frac{3r^3}{R^3}\right) \Leftrightarrow$$

$\sum_{cyc} \sin\frac{A}{2} - \sum_{cyc} \sin^2 A \sin\frac{B}{2} \le 3\left(\frac{1}{2} - \frac{3r^3}{R^3}\right)$ which follows from inequalities:

$\sum_{cyc} \sin\frac{A}{2} \le \frac{3}{2}$ and $\sum_{cyc} \sin^2 A \sin\frac{B}{2} \ge \frac{9r^3}{R^3}$

Lemma 1. 5. In ΔABC the following relationship holds:

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2}$$

R. Kooistra, Item 2.27, Geometric Inequality, O.Bottema,1969

Proof. Applying Jensen Inequality for concave function $x \to \cos x, x \in (0, \pi)$, we have:

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le 3\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3} = \frac{3}{2} \text{ equality if } A = B = C.$$

Equality holds if and only if triangle is equilateral.

Lemma 2. 6. In ΔABC the following relationship holds:

$$\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}} \le 2\left(\frac{R}{r} + 1\right)$$

Proof. From BCS inequality, we have:

$$\left(\sum_{cyc} \frac{1}{\sin\frac{A}{2}}\right)^2 = \left(\sum_{cyc} \frac{1}{\sqrt{\frac{(s-b)(s-c)}{bc}}}\right)^2 = \left(\sum_{cyc} \frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}}\right)^2 \le$$

$$\le \left(\sum_{cyc} bc\right)\left(\sum_{cyc} \frac{1}{(s-b)(s-c)}\right) = (s^2 + r^2 + 4Rr) \cdot \frac{1}{r^2} \overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le} (4R^2 + 4Rr + 3r^2 + r^2 + 4Rr) \cdot \frac{1}{r^2} = (4R^2 + 8Rr + 4r^2) \cdot \frac{1}{r^2} =$$

$= 4(R+r)^2 \cdot \frac{1}{r^2}$. Hence, $\sum_{cyc} \frac{1}{\sin\frac{A}{2}} \le 2\left(\frac{R}{r} + 1\right)$.

Equality holds if and only if triangle is equilateral.Now, let's prove that inequality:

$$\sum_{cyc} sin^2 A sin\frac{B}{2} \ge \frac{9r^3}{R^3}$$

Using Bergstrom inequality, we have:

$$\sum_{cyc} sin^2 A sin\frac{B}{2} = \sum_{cyc} \frac{sin^2 A}{\frac{1}{sin\frac{B}{2}}} \ge \frac{\left(\sum_{cyc} sinA\right)^2}{\sum_{cyc}\frac{1}{sin\frac{B}{2}}} = \frac{\left(\frac{s}{R}\right)^2}{\sum_{cyc}\frac{1}{sin\frac{A}{2}}} \overset{Lemma\ 2}{\ge} \frac{\left(\frac{s}{R}\right)^2}{2\left(\frac{R}{r}+1\right)} \overset{(2)}{\ge} \frac{9r^3}{R^3}$$

where $(2) \Leftrightarrow \frac{\left(\frac{s}{R}\right)^2}{2\left(\frac{R}{r}+1\right)} \ge \frac{9r^3}{R^3} \Leftrightarrow \frac{\frac{s^2}{R^2}}{\frac{2(R+r)}{r}} \ge \frac{9r^3}{R^3} \Leftrightarrow Rs^2 \ge 18r^2(R+r)$**, which follows from**

$s^2 \ge 16Rr - 5r^2$ **(Gerretsen). It remains to prove that:**

$$R(16Rr - 5r^2) \ge 18r^2(R+r) \Leftrightarrow 16R^2 - 23Rr - 18r^3 \ge 0 \Leftrightarrow$$

$(R-2r)(16R^2+9r^2) \ge 0$ **which is true from** $R \ge 2r$ **(Euler).**

Equality holds if and only if triangle is equilateral.

7. In acute ΔABC the following relationship holds:

$$\sum_{cyc} tan^2 A cot\frac{B}{2} \ge \frac{18\sqrt{3}r}{R}$$

Proposed by Marin Chirciu-Romania

Solution by proposer **Using Bergstrom inequality, we have:**

$$\sum_{cyc} tan^2 A cot\frac{B}{2} = \sum_{cyc} \frac{tan^2 A}{tan\frac{B}{2}} \ge \frac{\left(\sum_{cyc} tanA\right)^2}{\sum_{cyc} tan\frac{B}{2}} = \frac{\left(\frac{2sr}{s^2-(2R+r)^2}\right)^2}{\sum_{cyc} tan\frac{A}{2}} \overset{(1)}{\ge} \frac{\left(3\sqrt{3}\right)^2}{\frac{4R+r}{s}} =$$

$$= \frac{27s}{4R+r} \overset{(2)}{\ge} \frac{18\sqrt{3}r}{R}$$

Where $(1) \Leftrightarrow \frac{2sr}{s^2-(2R+r)^2} \ge 3\sqrt{3}$**, which follows from** $s \ge 3\sqrt{3}r$ **(Mitrinovic) and**

$s^2 \le 4R^2 + 4Rr + 3r^2$ **(Gerretsen). It follows that:**

$$\frac{2sr}{s^2-(2R+r)^2} \ge \frac{2r \cdot 3r\sqrt{3}}{4R^2+4Rr+3r^2-(2R+r)^2} = \frac{6r^2\sqrt{3}}{2r^2} = 3\sqrt{3} \Leftrightarrow$$

$$2r^2 4R^2 + 4Rr + 3r^2 - (2R+r)^2$$

$(2) \Leftrightarrow \frac{27s}{4R+r} \geq \frac{18\sqrt{3}r}{R} \Leftrightarrow 3Rs \geq 2\sqrt{3}r(4R+r)$, which follows from

$s \geq 3r\sqrt{3}(Mitrinovic)$. It remains to prove that:

$$3R \cdot 3\sqrt{3}r \geq 2\sqrt{3}r(4R+r) \Leftrightarrow 9R \geq 2(4R+r) \Leftrightarrow R \geq 2r(Euler).$$

Equality holds if and only if triangle is equilateral.

8. In acute ΔABC the following relationship holds:

$$\sum_{cyc} cot^2 A tan\frac{B}{2} \geq \frac{2r}{\sqrt{3}R}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using Bergstrom inequality, we have:

$$\sum_{cyc} cot^2 A tan\frac{B}{2} = \sum_{cyc} \frac{cot^2 A}{cot\frac{B}{2}} \geq \frac{\left(\sum_{cyc} cotA\right)^2}{\sum_{cyc} cot\frac{B}{2}} = \frac{\left(\frac{s^2-r^2-4Rr}{2sr}\right)^2}{\sum_{cyc} cot\frac{A}{2}} \overset{(1)}{\geq} \frac{\left(\sqrt{3}\right)^2}{\frac{s}{r}} = \frac{2r}{\sqrt{3}R}$$

Where $(1) \Leftrightarrow \frac{s^2-r^2-4Rr}{2sr} \geq \sqrt{3}$, which follows from $s \leq \frac{3\sqrt{3}R}{2}$ $(Mitrinovic)$ and

$$s^2 \geq 16Rr - 5r^2 \ (Gerretsen).$$

We get:

$$\frac{16Rr-5r^2-r^2-4Rr}{2sr} \geq \frac{16Rr-6r^2}{3\sqrt{3}Rr} \overset{(3)}{\geq} \sqrt{3}, \text{ where } (3) \Leftrightarrow \frac{16Rr-6r^2}{3\sqrt{3}Rr} \geq \sqrt{3} \Leftrightarrow$$

$$12Rr - 6r^2 \geq 9Rr \Leftrightarrow R \geq 2r \ (Euler).$$

$$(2) \Leftrightarrow \frac{3r}{s} \geq \frac{2r}{\sqrt{3}R} \Leftrightarrow s \leq \frac{3\sqrt{3}R}{2} \ (Mitrinovic).$$

Equality holds if and only if triangle is equilateral.

Reference:

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