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ABOUT AN INEQUALITY BY ERTAN YILDIRIM-III

By Marin Chirciu-Romania

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1) In ΔABC the following relationship holds:

$$4R \le \frac{1}{a+b+c}\sum bc\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right) \le \frac{2R^2}{r}$$

Proposed by Ertan Yildirim-Turkey

Solution by Marin Chirciu-Romania

Lemma. 2) In ΔABC the following relationship holds:

$$\sum bc\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right) = 16R^2rs\sum\frac{1}{a^2}$$

Proof. We have:

$$\sum bc\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right) = \sum bc\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\right) = \sum bc\left(\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}\right) =$$

$$= \sum bc\left(\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}}\right) = \sum bc\left(\frac{2}{2\sin\frac{A}{2}\cos\frac{A}{2}}\right) = \sum\frac{2bc}{\sin A} = \sum\frac{2bc}{\frac{a}{2R}} =$$

$$= 4R\sum\frac{bc}{a} = 4R\sum\frac{abc}{a^2} = 4R\cdot abc\sum\frac{1}{a^2} = 4R\cdot 4Rrs\sum\frac{1}{a^2} = 16R^2rs\sum\frac{1}{a^2}$$

Let's get back to the main problem.

Using lemma inequality can be write as:

$$4R \le \frac{1}{a+b+c}\cdot 16R^2rs\sum\frac{1}{a^2} \le \frac{2R^2}{r} \Leftrightarrow 4R \le \frac{1}{2s}\cdot 16R^2rs\sum\frac{1}{a^2} \le \frac{2R^2}{r} \Leftrightarrow$$

$$\frac{1}{2Rr} \le \sum\frac{1}{a^2} \le \frac{1}{4r^2}, (J.Steinig, 1963)$$

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\frac{16r^2(4R+r)^2}{3F} \le \sum bc\left(tan\frac{A}{2}+cot\frac{B}{2}\right) \le \frac{4R^2(4R+r)^2}{3F}$$

Proposed by Marin Chirciu-Romania

Solution.

Lemma. 4) In ΔABC the following relationship holds:

$$\sum bc\left(tan\frac{A}{2}+cot\frac{B}{2}\right) = \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F}$$

Proof. We have:

$$\sum bc\left(tan\frac{A}{2}+cot\frac{B}{2}\right) = \sum bc\left(\frac{sin\frac{A}{2}}{cos\frac{A}{2}}+\frac{cos\frac{A}{2}}{sin\frac{A}{2}}\right) = \sum bc\left(\frac{sin^2\frac{A}{2}+cos^2\frac{A}{2}}{sin\frac{A}{2}cos\frac{A}{2}}\right) =$$

$$= \sum bc\left(\frac{1}{sin\frac{A}{2}cos\frac{A}{2}}\right) = \sum bc\left(\frac{2}{2sin\frac{A}{2}cos\frac{A}{2}}\right) = \sum\frac{2bc}{sinA} = \sum\frac{2bc}{\frac{a}{2R}} =$$

$$= 4R\frac{\sum b^2c^2}{abc} = 4R\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs} =$$

$$= \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F}$$

Let's get back to the main problem.

For RHD, we have:

$$\sum bc\left(tan\frac{A}{2}+cot\frac{B}{2}\right) = \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F} \le$$

$$\le \frac{\frac{R(4R+r)^2}{2(2R-r)}(4R^2+4Rr+3r^2+2r^2-8Rr)+r^2(4R+r)^2}{F} =$$

$$= \frac{(4R+r)^2(4R^3-4R^2r+9Rr^2-2r^3)}{2F(2R-r)} \overset{(1)}{\le} \frac{(4R+r)^2}{2F}\cdot\frac{8R^2}{3} = \frac{4R^2(4R+r)^2}{3F}$$

where $(1) \Leftrightarrow \frac{4R^3-4R^2r+9Rr^2-2r^3}{2R-r} \le \frac{8R^2}{3} \Leftrightarrow 4R^3+4R^2r-27Rr^2+6r^3 \ge 0 \Leftrightarrow$

$(R-2r)(4R^2+12Rr-3r^2) \ge 0$ **which is true from** $R \ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

For LHS, we have:

$$\sum bc\left(tan\frac{A}{2}+cot\frac{B}{2}\right)=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{F}\geq$$

$$\geq\frac{\frac{r(4R+r)^2}{R+r}(16Rr-5r^2+2r^2-8Rr)+r^2(4R+r)^2}{F}=$$

$$=\frac{r^2(4R+r)^2(9R-2r)}{F(R+r)}\overset{(2)}{\geq}\frac{r^2(4R+r)^2}{F}\cdot\frac{16}{3}=\frac{16r^2(4R+r)^2}{3F}$$

where $(2)\Leftrightarrow\frac{9R-2r}{R+r}\leq\frac{16}{3}\Leftrightarrow 11R\geq 22r\Leftrightarrow R\geq 2r(Euler)$

Equality holds if and only if triangle is equilateral.

Well-known Blundon-Gerretsen inequality:

$$\frac{r(4R+r)^2}{R+r}\leq 16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

Remark. In same class of problems.

4) In ΔABC the following relationship holds:

$$\frac{2}{r}\leq\sum\frac{1}{a}\left(tan\frac{A}{2}+cot\frac{A}{2}\right)\leq\frac{R}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution.

Lemma. 6) In ΔABC the following relationship holds:

$$\sum\frac{1}{a}\left(tan\frac{A}{2}+cot\frac{A}{2}\right)=4R\sum\frac{1}{a^2}$$

Proof. We have:

$$\sum\frac{1}{a}\left(tan\frac{A}{2}+cot\frac{A}{2}\right)=\sum\frac{1}{a}\left(\frac{sin\frac{A}{2}}{cos\frac{A}{2}}+\frac{cos\frac{A}{2}}{sin\frac{A}{2}}\right)=\sum\frac{1}{a}\left(\frac{sin^2\frac{A}{2}+cos^2\frac{A}{2}}{sin\frac{A}{2}cos\frac{A}{2}}\right)=$$

$$=\sum\frac{1}{a}\left(\frac{1}{sin\frac{A}{2}cos\frac{A}{2}}\right)=\sum\frac{1}{a}\left(\frac{2}{2sin\frac{A}{2}cos\frac{A}{2}}\right)=\sum\frac{1}{a}\cdot\frac{2}{sinA}=\sum\frac{1}{a}\cdot\frac{2}{\frac{a}{2R}}$$

$$=4R\sum\frac{1}{a^2}$$

Let's get back to the main problem.

Using Lemma inequality written as:

$$\frac{2}{r} \le 4R\sum \frac{1}{a^2} \le \frac{R}{r^2} \Leftrightarrow \frac{1}{2Rr} \le \sum \frac{1}{a^2} \le \frac{1}{4r^2}, (J.Steinig, 1963)$$

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$24Rr\sqrt{3} \le \sum a^2\left(tan\frac{A}{2} + cot\frac{A}{2}\right) \le 12R^2\sqrt{3}$$

Proposed by Marin Chirciu-Romania

Solution.

Lemma 8) In ΔABC the following relationship holds:

$$\sum a^2\left(tan\frac{A}{2} + cot\frac{A}{2}\right) = 8Rs$$

Proof. We have:

$$\sum a^2\left(tan\frac{A}{2} + cot\frac{A}{2}\right) = \sum a^2\left(\frac{sin\frac{A}{2}}{cos\frac{A}{2}} + \frac{cos\frac{A}{2}}{sin\frac{A}{2}}\right) = \sum a^2\left(\frac{1}{sin\frac{A}{2}cos\frac{A}{2}}\right) =$$

$$= \sum a^2\left(\frac{2}{2sin\frac{A}{2}cos\frac{A}{2}}\right) = \sum a^2 \cdot \frac{2}{sinA} = \sum a^2 \cdot \frac{2}{\frac{a}{2R}} = 4R\sum a = 4R \cdot 2s$$

$$= 8Rs$$

Let's get back to the main problem.

Using Lemma and $3r\sqrt{3} \le s \le \frac{3R\sqrt{3}}{2}$ ($Mitrinovic$) we get conclusion.

Equality holds if and only if triangle is equilateral.

9) In ΔABC the following relationship holds:

$$\frac{2}{R} \le \sum \frac{1}{b+c}\left(tan\frac{A}{2} + cot\frac{A}{2}\right) \le \frac{1}{r}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 10) In ΔABC the following relationship holds:**

$$\sum \frac{1}{b+c}\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)=\frac{s^2(s^2+r^2+16Rr)+r^2(4R+r)^2}{2rs^2(s^2+r^2+2Rr)}$$

Proof. We have:

$$\sum \frac{1}{b+c}\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)=\sum \frac{1}{b+c}\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\right)=$$

$$=\sum \frac{1}{b+c}\left(\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)=\sum \frac{1}{b+c}\left(\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)=$$

$$=\sum \frac{1}{b+c}\left(\frac{2}{2\sin\frac{A}{2}\cos\frac{A}{2}}\right)=\sum \frac{1}{b+c}\cdot\frac{2}{\sin A}=\sum \frac{1}{b+c}\cdot\frac{2}{\frac{a}{2R}}=$$

$$=4R\sum \frac{1}{a(b+c)}=4R\cdot\frac{s^2(s^2+r^2+16Rr)+r^2(4R+r)^2}{8Rrs^2(s^2+r^2+2Rr)}=$$

$$=\frac{s^2(s^2+r^2+16Rr)+r^2(4R+r)^2}{2rs^2(s^2+r^2+2Rr)}$$

which follows from:

$$\sum \frac{1}{a(b+c)}=\frac{\sum bc(a+b)(a+c)}{abc\prod(b+c)}=\frac{s^2(s^2+r^2+16Rr)+r^2(4R+r)^2}{4Rrs\cdot 2s(s^2+r^2+2Rr)}=$$

$$=\frac{s^2(s^2+r^2+16Rr)+r^2(4R+r)^2}{8Rrs^2(s^2+r^2+2Rr)}$$

Hence,

$$\sum \frac{1}{a(b+c)}=\frac{s^2(s^2+r^2+16Rr)+r^2(4R+r)^2}{8Rrs^2(s^2+r^2+2Rr)}$$

$$\sum bc(a+b)(a+c)=s^2(s^2+r^2+16Rr)+r^2(4R+r)^2$$

Let's get back to the main problem.

For RHD, we have:

$$\frac{s^2(s^2+r^2+16Rr)+r^2(4R+r)^2}{2rs^2(s^2+r^2+2Rr)}\le\frac{1}{r}\Leftrightarrow s^2(s^2-12Rr)\ge r^2(4R+r)^2$$

which follows from $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}(Gerretsen)$

It remains to prove that: $\frac{r(4R+r)^2}{R+r}(16Rr-5r^2-12Rr) \geq r^2(4R+r)^2 \Leftrightarrow$

$\Leftrightarrow R \geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

For LHS, we have:

$$\frac{s^2(s^2+r^2+16Rr)+r^2(4R+r)^2}{2rs^2(s^2+r^2+2Rr)} \geq \frac{2}{R} \Leftrightarrow$$

$$Rs^2(s^2+r^2+16Rr)+Rr^2(4R+r)^2 \geq 4rs^2(s^2+r^2+2Rr) \Leftrightarrow$$

$$s^2[s^2(R-4r)+r(16R^2-6Rr-4r^2)]+Rr^2(4R+r)^2 \geq 0$$

Distinguish cases:

Case 1) If $(R-4r) \geq 0$ inequality is obviously.

Case 2) If $(R-4r) < 0$ inequality rewritten as:

$Rr^2(4R+r)^2 \geq s^2[s^2(4r-R)-r(16R^2-6Rr-4r^2)]$, which follows from:

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2$$

It remains to prove that:

$$Rr^2(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}[(4R^2+4Rr+3r^2)(4r-R)-r(16r^2-6Rr-4r^2)] \geq 0$$

$\Leftrightarrow 4R^3+4R^2r-15Rr^2-18r^3 \geq 0 \Leftrightarrow (R-2r)(2R+3r)^2 \geq 0$ true from

$R \geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

11) In ΔABC the following relationship holds:

$$64rs \leq \sum(b+c)^2\left(tan\frac{B}{2}+tan\frac{C}{2}\right) \leq \frac{16R^2s}{r}$$

Proposed by Mari Chirciu-Romania

Solution. **Lemma. 12) In ΔABC the following relationship holds:**

$$\sum(b+c)^2\left(tan\frac{B}{2}+tan\frac{C}{2}\right) = \frac{4s(s^2+r^26Rr)}{r}$$

Proof. We have:

$$\sum(b+c)^2\left(tan\frac{B}{2}+tan\frac{C}{2}\right) = \sum(b+c)^2\left(\frac{sin\frac{A}{2}}{cos\frac{A}{2}}+\frac{cos\frac{A}{2}}{sin\frac{A}{2}}\right) =$$

$$=\sum(b+c)^2\left(\frac{sin^2\frac{A}{2}+cos^2\frac{A}{2}}{sin\frac{A}{2}cos\frac{A}{2}}\right)=\sum(b+c)^2\left(\frac{1}{sin\frac{A}{2}cos\frac{A}{2}}\right)=$$

$$=\sum(b+c)^2\left(\frac{2}{2sin\frac{A}{2}cos\frac{A}{2}}\right)=\sum(b+c)^2\cdot\frac{2}{sinA}=\sum(b+c)^2\cdot\frac{2}{\frac{a}{2R}}=$$

$$=4R\sum\frac{(b+c)^2}{a}=4R\cdot\frac{s(s^2+r^2-6Rr)}{Rr}=\frac{4s(s^2+r^2-6Rr)}{r}$$

which follows from:

$$\sum\frac{(b+c)^2}{a}=\frac{\sum bc(b+c)^2}{abc}=\frac{4s^2(s^2+r^2-6Rr)}{4Rrs}=\frac{s(s^2+r^2-6Rr)}{Rr}$$

Hence,

$$\sum\frac{(b+c)^2}{a}=\frac{s(s^2+r^2-6Rr)}{Rr}$$

Let's get back to the main problem.For RHD, using Lemma we have:

$$\frac{4s(s^2+r^2-6Rr)}{r}\le\frac{16R^2}{r}\Leftrightarrow s^2\le 4R^2+6Rr-r^2$$

which follows from $s^2\le 4R^2+4Rr+3r^2(Gerretsen)$

Equality holds if and only if triangle is equilateral.For LHS, using Lemma we have:

$$\frac{4s(s^2+r^2-6Rr)}{r}\ge 64rs\Leftrightarrow s^2\ge 6Rr+15r^2$$

which folows from $s^2\ge 16Rr-5r^2(Gerretsen)$

It remains to prove $16Rr-5r^2\ge 6Rr+15r^2\Leftrightarrow R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

13) In ΔABC the following relationship holds:

$$\frac{6}{Rs}\le\sum\frac{1}{bc}\left(tan\frac{B}{2}+tan\frac{C}{2}\right)\le\frac{3}{rs}$$

Proposed by Marin Chirciu-Romania

Solution.**Lemma. 14) In ΔABC the following relationship holds:**

$$\sum\frac{1}{bc}\left(tan\frac{B}{2}+tan\frac{C}{2}\right)=\frac{3}{sr}$$

Proof. We have:

$$\sum \frac{1}{bc}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\sum \frac{1}{bc}\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\right)=\sum \frac{1}{bc}\left(\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)$$

$$=\sum \frac{1}{bc}\left(\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)=\sum \frac{1}{bc}\left(\frac{2}{2\sin\frac{A}{2}\cos\frac{A}{2}}\right)=\sum \frac{1}{bc}\cdot\frac{2}{\sin A}=$$

$$=\sum \frac{1}{bc}\cdot\frac{2}{\frac{a}{2R}}=4R\sum\frac{1}{abc}=4R\cdot\frac{3}{4Rrs}=\frac{3}{sr}$$

Let's get back to the main problem.Using Lemma and $R\geq 2r(Euler)$ we get conclusion.

15) In ΔABC the following relationship holds:

$$24r\leq\sum a\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)\leq 12R$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 16) In ΔABC the following relationship holds:**

$$\sum a\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)=12R$$

Proof. We have:

$$\sum a\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)=\sum a\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\right)=\sum a\left(\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)=$$

$=\sum a\left(\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)=\sum a\left(\frac{2}{2\sin\frac{A}{2}\cos\frac{A}{2}}\right)=\sum a\cdot\frac{2}{\sin A}=\sum a\cdot\frac{2}{\frac{a}{2R}}=12R$**.Let's get back to the main problem.Using Lemma and $R\geq 2r(Euler)$ we get conclusion.**

Equality in Lhs if and only if triangle is equilateral.

17) In ΔABC the following relationship holds:

$$48r\leq\sum(b+c)\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)\leq\frac{12R^2}{r}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 18) In ΔABC the following relationship holds:**

$$\sum(b+c)\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)=\frac{2(s^2+r^2-2Rr)}{r}$$

Proof. We have:

$$\sum(b+c)\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)=\sum(b+c)\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}\right)=$$

$$=\sum(b+c)\left(\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)=\sum(b+c)\left(\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}}\right)=$$

$$=\sum(b+c)\left(\frac{2}{2\sin\frac{A}{2}\cos\frac{A}{2}}\right)=\sum(b+c)\cdot\frac{2}{\sin A}=\sum(b+c)\cdot\frac{2}{\frac{a}{2R}}=$$

$$=4R\sum\frac{b+c}{a}=4R\cdot\frac{s^2+r^2-2Rr}{2Rr}=\frac{2(s^2+r^2-2Rr)}{r}$$

Let's get back to the main problem.

For RHD, using Lemma we have:

$\frac{2(s^2+r^2-2Rr)}{r}\le\frac{12R^2}{r}\Leftrightarrow s^2\le 6R^2+2Rr-r^2$ **which follows from**

$s^2\le 4R^2+4Rr+3r^2(Gerretsen)$, **it remains to prove that:**

$4R^2+4Rr+3r^2\le 6R^2+2Rr-r^2\Leftrightarrow R^2-Rr-2r^2\ge 0\Leftrightarrow$

$(R-2r)(R+r)\ge 0$ **which is true from** $R\ge 2r(Euler)$.

For LHS, using Lemma we have:

$\frac{2(s^2+r^2-2Rr)}{r}\ge 48r\Leftrightarrow s^2\ge 2Rr+23r^2$ **which follows from**

$s^2\ge 16Rr-5r^2(Gerretsen)$ **it remains to prove:**

$16Rr-5r^2\ge 2Rr+23r^2\Leftrightarrow R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

Reference:

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