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ABOUT AN INEQUALITY BY DRAGOLJUB MILOSEVIC-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\sum \frac{h_a^2}{r_a} \geq 4r\left(2-\frac{r}{R}\right)^2$$

Proposed by Dragoljub Milosevic – Serbia

Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\sum \frac{h_a^2}{r_a} = \frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4R^2r}$$

Proof.

Using $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum \frac{h_a^2}{r_a} = \sum \frac{\left(\frac{2S}{a}\right)^2}{\frac{S}{s-a}} = 4S\sum \frac{s-a}{a^2} = \frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4R^2r},$$ *which follows from:*

$$\sum \frac{s-a}{a^2} = \frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s}$$

Let's get back to the main problem:

Using the Lemma we write the inequality:

$$\frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4R^2r} \geq 4r\left(2-\frac{r}{R}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow s^2(s^2+2r^2-12Rr)+r^3(4R+r) \geq 16r^2(2R-r)^2,$$

Which follows from Gerretsen's inequality: $s^2 \geq 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^3(4R+r) \geq 16r^2(2R-r)^2 \Leftrightarrow$$

$$\Leftrightarrow (2R-r)^2 \geq (2R-r)^2$$ *obviously with equality.*

Equality holds if and only if the triangle is equilateral.

Remark. *Let's emphasize and inequality having an opposite sense.*

3)In ΔABC the following relationship holds:

$$\sum \frac{h_a^2}{r_a} \le \frac{1}{r}(2R-r)^2$$

Proposed by Marin Chirciu – Romania

Solution *Using the Lemma we write the inequality:*

$$\frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s} \le \frac{1}{r}(2R-r)^2 \Leftrightarrow$$

$$\Leftrightarrow s^2(s^2+2r^2-12Rr)+r^3(4R+r) \le 4r\text{^}2(2R-r)^2,$$

which follows from Gerretsen's: $s^2 \le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^3(4R+r) \le 4r^2(2R-r)^2 \Leftrightarrow$$

$\Leftrightarrow 4R^2r^2 \ge 16r^4$, *which follows from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *We can write the double inequality:*

4) In ΔABC the following relationship holds:

$$4r\left(2-\frac{r}{R}\right)^2 \le \sum \frac{h_a^2}{r_a} \le \frac{1}{r}(2R-r)^2$$

Solution *See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.*

Remark. *Switching between them* h_a *and* r_a *we propose:*

5) In ΔABC the following relationship holds:

$$\frac{1}{r}(2R-r)^2 \le \sum \frac{r_a^2}{h_a} \le \frac{9R^4}{16r^3}$$

Proposed by Marin Chirciu – Romania

Solution *We prove the following lemma:*

Lemma.

6) In ΔABC the following relationship holds:

$$\sum \frac{r_a^2}{h_a} = \frac{8R^2+2Rr-s^2}{r}$$

Proof. *Using* $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum \frac{r_a^2}{h_a} = \sum \frac{\left(\frac{S}{s-a}\right)^2}{\frac{2S}{a}} = \frac{S}{2}\sum \frac{a}{(s-a)^2} = \frac{8R^2+2Rr-s^2}{r},$$ *which follows from:*

$$\sum \frac{a}{(s-a)^2} = \frac{2(8R^2+2Rr-s^2)}{r^2 s}$$

Let's get back to the main problem: LHS inequality:

Using Lemma, LHS inequality can be written:

$$\frac{8R^2+2Rr-s^2}{r} \ge \frac{1}{r}(2R-r)^2 \Leftrightarrow s^2 \le 4R^2+6Rr-r^2$$

which follows from Gerretsen's inequality: $s^2 \le 4R^2+4Rr+3r^2$

It remains to prove that:

$$4R^2+4Rr+3r^2 \le 4R^2+6Rr-r^2 \Leftrightarrow R \ge 2r \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral

RHS inequality: Using Lemma the RHS inequality can be written:

$$\frac{8R^2+2Rr-s^2}{r} \le \frac{9R^4}{16r^3} \Leftrightarrow 16r^2(8R^2+2Rr-s^2) \le 9R^4$$

which follows from Gerretsen's inequality: $s^2 \ge 16Rr-5r^2$.

It remains to prove that:

$$16r^2\left(8R^2+2Rr-(16Rr-5r^2)\right) \le 9R^4 \Leftrightarrow 9R^4-128R^2r^2+224Rr^3-80r^4 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)^2(9R^2+36Rr-20r^2) \ge 0, \text{ obviously.}$$

Equality holds if and only if the triangle is equilateral.

Remark. *Between the sums* $\sum \frac{h_a^2}{r_a}$ *and* $\sum \frac{r_a^2}{h_a}$ *we can write the relationship:*

7) In ΔABC the following relationship holds:

$$\sum \frac{h_a^2}{r_a} \le \sum \frac{r_a^2}{h_a}$$

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Solution *Using the sums:*

$$\sum \frac{h_a^2}{r_a} = \frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4R^2r}$$ *and* $$\sum \frac{r_a^2}{h_a} = \frac{8R^2+2Rr-s^2}{r}$$ *the inequality can be written:*

$$\frac{s^4+s^2(2r^2-12Rr)+r^3(4R+r)}{4R^2r}\le\frac{8R^2+2Rr-s^2}{r}\Leftrightarrow$$

$$\Leftrightarrow s^2(s^2+2r^2-12Rr+4R^2)+r^3(4R+r)\le 8R^3(4R+r),$$

which follows from Gerretsen's inequality: $s^2\le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr+4R^2)+r^3(4R+r)\le$$

$$\le 8R^3(4R+r)\Leftrightarrow 2R^3-3R^2r-4r^3\ge 0\Leftrightarrow (R-2r)(2R^2+Rr+2r^2)\ge 0$$

which follows from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *We can write the sequence of inequalities:*

8) In ΔABC the following inequality holds:

$$4r\left(2-\frac{r}{R}\right)^2\le\sum\frac{h_a^2}{r_a}\le\frac{1}{r}(2R-r)^2\le\sum\frac{r_a^2}{h_a}\le\frac{9R^4}{16r^3}$$

Solution *See inequalities 4) and 5). Equality holds if and only if the triangle is equilateral.*

Reference:

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